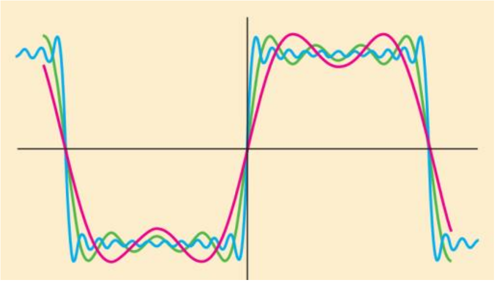

Fall 2019

微分方程
Differential Equations

Unit 11.1
Orthogonal Functions

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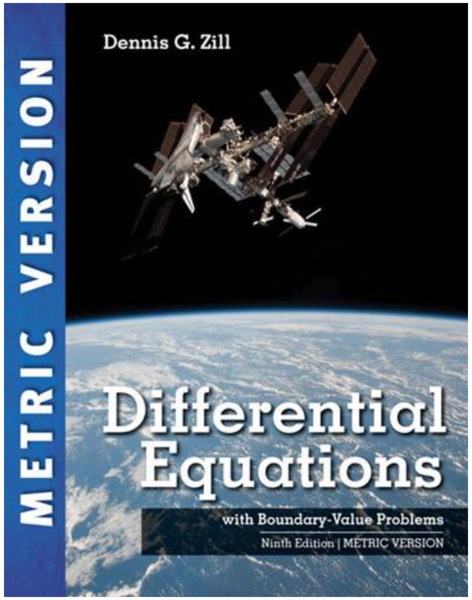

$$\left(f_1,\ f_2 \right) = \int_a^b f_1(x) \cdot f_2(x)\, dx = 0$$

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- In R^3 vector space

$$x = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad y = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad x_i, \; y_i \in R$$

- Inner Product (Dot Product) of x and y in R^3:

$$(\,\cdot\,,\,\cdot\,) : R^3 \times R^3 \rightarrow R$$

$$(\,x\,,\,y\,) =$$

11.1: Properties of Inner Product

$$x,\ y,\ z \in R^3, \qquad k \in R$$

(1) $(x, y) = (y, x)$

(2) $(kx, y) = k(x, y) = (x, ky)$

(3) $(x, x) = 0$ only if $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$(x, x) > 0$ only if $x \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(4) $(x + y, z) = (x, z) + (y, z)$

$f_1, \; f_2,$: two functions on an interval $[a, b]$

$$\left(f_1, \; f_2 \right) \quad \triangleq \quad \int_a^b f_1(x) \cdot f_2(x) \, dx$$

(1) $\quad (\, f_1 \, , \, f_2 \,) = (\, f_2 \, , \, f_1 \,)$

(2) $\quad (\, k \, f_1 \, , \, f_2 \,) = k \, (\, f_1 \, , \, f_2 \,) = (\, f_1 \, , \, k \, f_2 \,)$

(3) $\quad (\, f_1 \, , \, f_1 \,) = 0$ only if $f_1 \equiv 0, \;\; \forall x \in [a, b]$

$(\, f_1 \, , \, f_1 \,) > 0$ only if $f_1 \not\equiv 0, \;\; \forall x \in [a, b]$

(4) $\quad (\, f_1 + f_2, \; f_3 \,) = (\, f_1 \, , \, f_3 \,) + (\, f_2 \, , \, f_3 \,)$

f_1, f_2, : two functions on an interval $[a, b]$

IF $\quad \left(f_1, \; f_2 \right) = \int_a^b f_1(x) \cdot f_2(x) \, dx = 0$

THEN $\quad f_1$, f_2 is said to be orthogonal on $[a, b]$

11.1: Example 1: Orthogonal Functions

(a) $f_1(x) = x^2, \quad f_2(x) = x^3, \quad$ on $[-1, 1]$

(b) $f_1(x) = x^2, \quad f_2(x) = x^4, \quad$ on $[-1, 1]$

$\left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}$:

a set of real-valued functions on $[a, b]$

IF $\quad \Big(\phi_m(x),\ \phi_n(x) \Big) = \int_a^b \phi_m(x) \cdot \phi_n(x)\, dx = 0, \quad m \neq n$

THEN it is an orthogonal set on $[a, b]$

$$\left\{1,\ \cos(x),\ \cos(2x),\ \cdots,\ \cos(nx),\ \cdots\right\} \qquad \text{on } I = [-\pi, \pi]$$

11.1: Definition of Norm

- Norm of $\begin{cases} \text{a vector} & \text{in } R^n \\ \text{a function} & \text{on } [a,b] \end{cases}$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in R^3$$

$f(x)$ a function on $[a,b]$

$$||u|| \triangleq$$

$$||f(x)|| \triangleq$$

11.1: Orthonormal Set

$$\left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}: \quad \text{on } [a,b]$$

(1) $$\left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}:$$

(2) $$||\phi_n(x)|| \ = \qquad , \quad n = 0, 1, 2, \cdots$$

i.e., $$(\phi_i(x),\ \phi_j(x)) \ = \begin{cases} & \text{if } i = j \\ & \text{if } i \neq j \end{cases}$$

11.1: Examples 3 & 4: Norms & Orthonormal Set

$$\left\{1,\ \cos(x),\ \cos(2x),\ \cdots,\ \cos(nx),\ \cdots\right\} \quad \text{on } I = [-\pi, \pi]$$

u in R^3

$u \;=$

$y = f(x)$ on $[a, b]$

$f(x) \quad =$

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b f(x) \cdot \phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \, dx}$$

$$= \frac{\Big(f(x), \ \phi_n(x)\Big)}{\Big(\phi_n(x), \ \phi_n(x)\Big)}$$

$$\Big\| \phi_n(x) \Big\|^2$$

11.1: Def 11.1.4: Orthogonal Set/Weight Functions

$$\left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}:$$

orthogonal with respect to a weight function $w(x)$

IF $\displaystyle\int_a^b w(x) \cdot \phi_m(x) \cdot \phi_n(x)\, dx = 0, \quad m \neq n$

$$f(x) = c_0\, \phi_0(x) + c_1\, \phi_1(x) + c_2\, \phi_2(x) + \cdots$$

$$c_n = \frac{\int_a^b f(x) \cdot w(x) \cdot \phi_n(x)\, dx}{\int_a^b w(x) \cdot \phi_n^2(x)\, dx}$$

11.1: Complete Sets

$$S = \left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}:$$

IF the ONLY function orthogonal to $\phi_i(x)$ is ZERO function,

THEN S is a complete set

$$\left\{ 1,\ \cos(x),\ \cos(2x),\ \cdots,\ \right\}$$

$$\left\{ 1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x),\ \cdots,\ \right\}$$

Summary - 11.1: Orthogonal Functions

f_1, f_2, : two functions on an interval $[a, b]$

- Inner Product of Functions

$$\left(f_1, f_2 \right) \triangleq \int_a^b f_1(x) \cdot f_2(x)\, dx$$

- Orthogonal Functions

$$\left(f_1, f_2 \right) = \int_a^b f_1(x) \cdot f_2(x)\, dx = 0$$

- Orthogonal Set $\left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}$:

IF $\left(\phi_m(x),\ \phi_n(x) \right) = \int_a^b \phi_m(x) \cdot \phi_n(x)\, dx = 0, \quad m \neq n$

- Orthonormal Set $\left\{ \phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots \right\}$:

i.e., $$(\phi_i(x),\ \phi_j(x)) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Summary - 11.1: Vector Decomposition

- Vector Decomposition & Orthogonal Series Expansion

$y = f(x)$ on $[a, b]$

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b f(x) \cdot \phi_n(x)\, dx}{\int_a^b \phi_n^2(x)\, dx} = \frac{\big(f(x),\ \phi_n(x)\big)}{\big(\phi_n(x),\ \phi_n(x)\big)}$$

$$\left\|\phi_n(x)\right\|^2$$

- Orthogonal Set/Weight Functions

$\left\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \cdots\right\}$:

orthogonal with respect to a weight function $w(x)$

IF $\displaystyle\int_a^b w(x) \cdot \phi_m(x) \cdot \phi_n(x)\, dx = 0, \quad m \neq n$

$$f(x) = c_0\, \phi_0(x) + c_1\, \phi_1(x) + c_2\, \phi_2(x) + \cdots$$

$$c_n = \frac{\int_a^b f(x) \cdot w(x) \cdot \phi_n(x)\, dx}{\int_a^b w(x) \cdot \phi_n^2(x)\, dx}$$