Fall 2019

微分方程
Differential Equations

Unit 07.6
Systems of Linear Differential Equations

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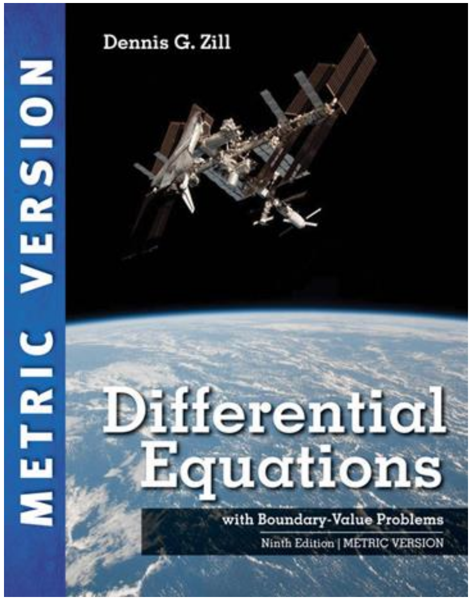
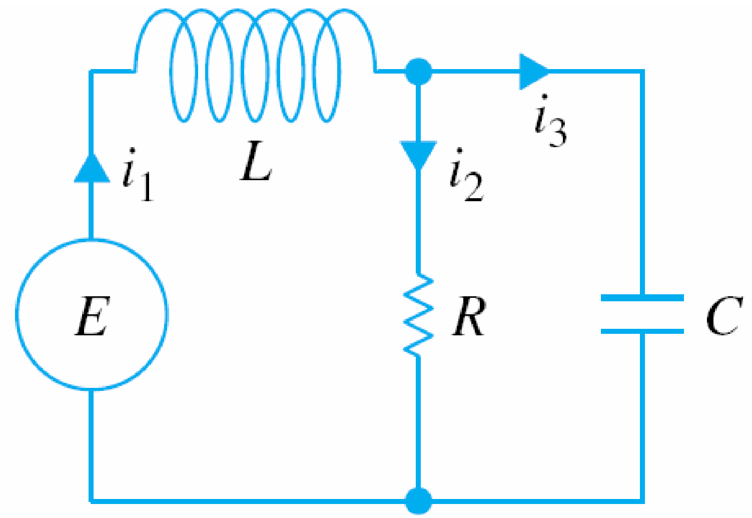
$$\begin{cases} L\,\frac{di_1}{dt} + R\,i_2 = E(t) \\ RC\,\frac{di_2}{dt} + i_2 - i_1 = 0 \end{cases}$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline



$$\Rightarrow \begin{cases} L\,\dfrac{di_1}{dt} + R\,i_2 = E(t) \\ RC\,\dfrac{di_2}{dt} + i_2 - i_1 = 0 \end{cases}$$

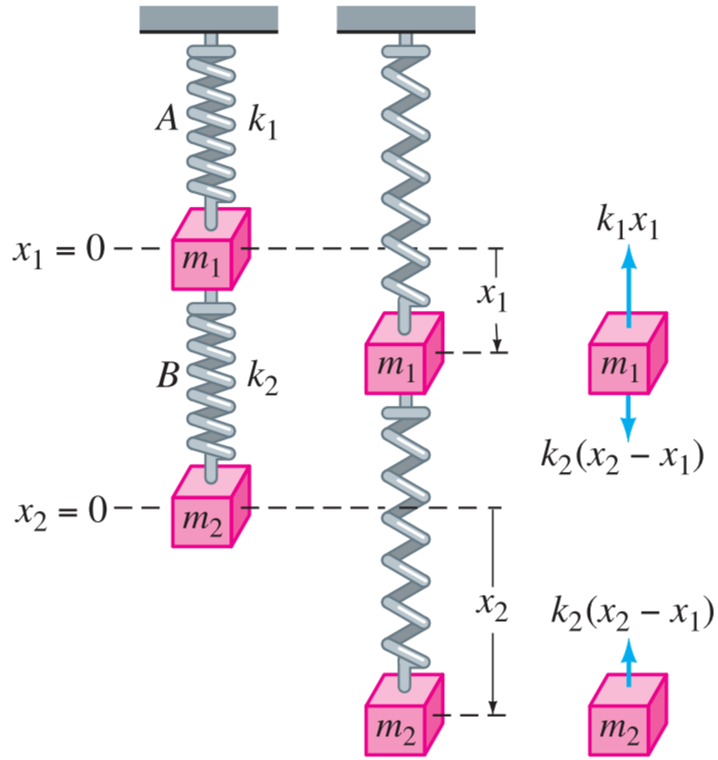
$$i_1(0) = 0, \quad i_2(0) = 0$$

$$E = 60V, \quad L = 1h,$$

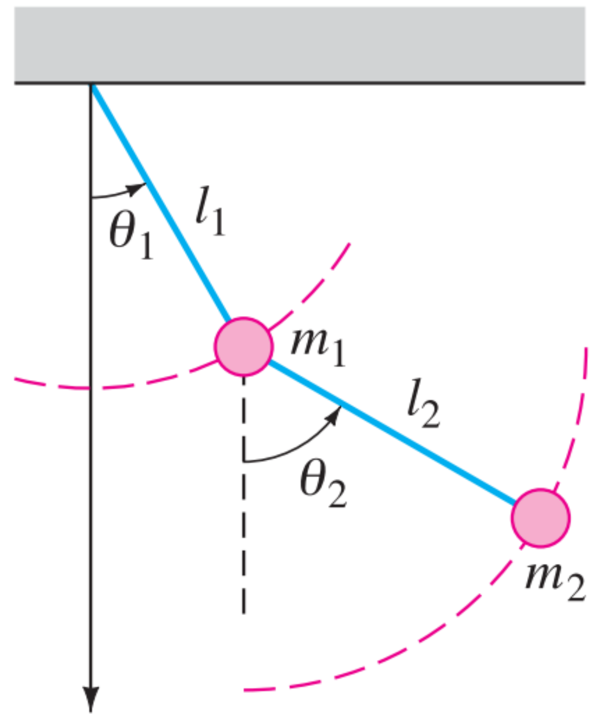
$$R = 50\Omega, \quad C = 10^{-4}f$$

By Laplace Transform

7.6: Systems of Linear Equations: Spring/Mass & Pendulum



(a) equilibrium **(b)** motion **(c)** forces



$$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1).$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1).$$

$$(m_1 + m_2)l_1^2\theta_1'' + m_2l_1l_2\theta_2'' \cos(\theta_1 - \theta_2) + m_2l_1l_2(\theta_2')^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)l_1g \sin\theta_1 = 0$$

$$m_2l_2^2\theta_2'' + m_2l_1l_2\theta_1'' \cos(\theta_1 - \theta_2) - m_2l_1l_2(\theta_1')^2 \sin(\theta_1 - \theta_2) + m_2l_2g \sin\theta_2 = 0.$$

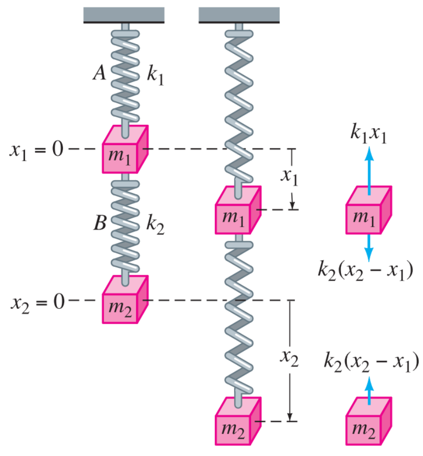
$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1)$$

$$m_2x_2'' = -k_2(x_2 - x_1).$$

$$(m_1 + m_2)l_1^2\theta_1'' + m_2l_1l_2\theta_2'' + (m_1 + m_2)l_1g\theta_1 = 0$$

$$m_2l_2^2\theta_2'' + m_2l_1l_2\theta_1'' + m_2l_2g\theta_2 = 0.$$

Summary - 7.6: Systems of Linear Equations



(a) equilibrium **(b)** motion **(c)** forces

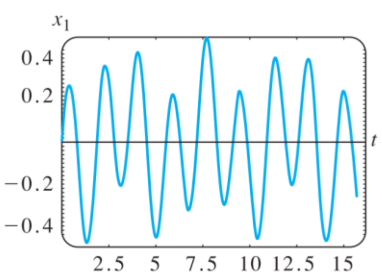
$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1).$$

Solve

$$\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}$$

subject to $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$.

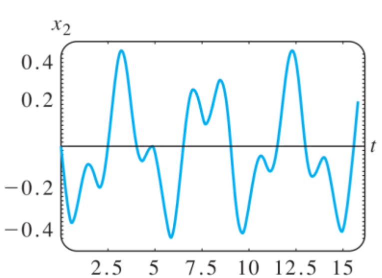
$$s^2X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) = 0$$
$$-4X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) = 0,$$



(a) plot of $x_1(t)$ vs. t

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12},$$

$$x_1(t) = -\frac{1}{5\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{6}{5\sqrt{12}}\mathscr{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$
$$= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t.$$

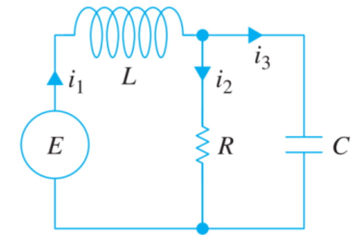


(b) plot of $x_2(t)$ vs. t

$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$x_2(t) = -\frac{2}{5\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} - \frac{3}{5\sqrt{12}}\mathscr{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$
$$= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t.$$

$$L\frac{di_1}{dt} + Ri_2 = E(t)$$
$$RC\frac{di_2}{dt} + i_2 - i_1 = 0.$$



$$\frac{di_1}{dt} + 50i_2 = 60$$
$$50(10^{-4})\frac{di_2}{dt} + i_2 - i_1 = 0$$

$$i_1(0) = 0,\ i_2(0) = 0.$$

$$sI_1(s) + \qquad 50I_2(s) = \frac{60}{s}$$
$$-200I_1(s) + (s+200)I_2(s) = 0,$$

$$I_1(s) = \frac{60s + 12{,}000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{60}{(s+100)^2}$$
$$I_2(s) = \frac{12{,}000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}.$$

$$i_1(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}$$
$$i_2(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}.$$