#### Fall 2019

# 微分方程 **Differential Equations**

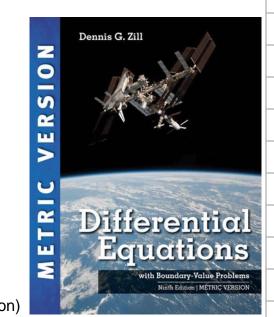
## Unit 07.6 Systems of Linear Differential Equations

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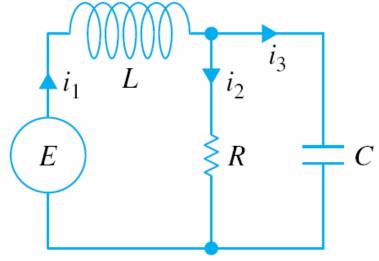
**NTU-EE** 

$$\begin{cases} L \frac{di_1}{dt} + R i_2 &= E(t) & \text{Sep19} - \text{Jan20} \\ RC \frac{di_2}{dt} + i_2 - i_1 &= 0 \end{cases}$$

Figures and images used in these lecture notes are adopted from Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)



- 7.1: Definition of Laplace Transform
- 7.2: Inverse Transforms and Transforms of Derivatives
  - 7.2.1: Inverse Transforms
  - 7.2.2: Transforms of Derivatives
- 7.3: Operational Properties I
  - 7.3.1: Translation on the s-Axis
  - 7.3.2: Translation on the t-Axis
  - 7.4: Operational Properties II
    - 7.4.1: Derivatives of a Transform
    - 7.4.2: Transforms of Integrals
    - 7.4.3: Transform of a Periodic Function
- 7.5: The Dirac Delta Function
- 7.6: Systems of Linear Differential Equations



$$\Rightarrow \begin{cases} L \frac{di_{1}}{dt} + R i_{2} &= E(t) \\ RC \frac{di_{2}}{dt} + i_{2} - i_{1} &= 0 \end{cases}$$

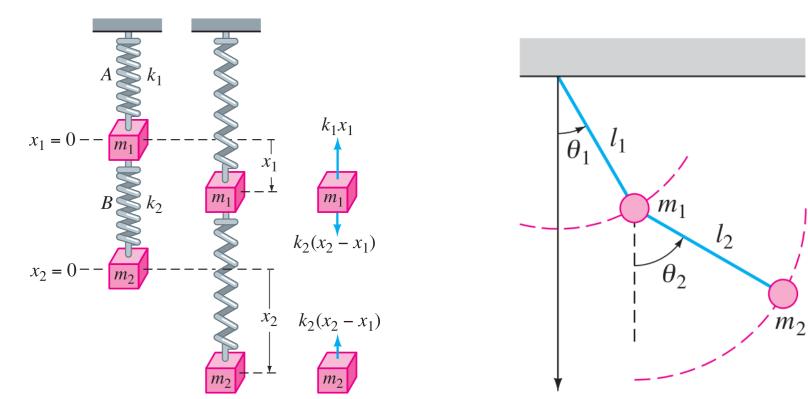
$$i_{1}(0) = 0, \quad i_{2}(0) = 0$$

E = 60V, L = 1h,

$$R = 50\Omega, \ C = 10^{-4}f$$

#### 7.6: Systems of Linear Equations: Spring/Mass & Pendulum

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$$m_{1}\frac{d^{2}x_{1}}{dt^{2}} = -k_{1}x_{1} + k_{2}(x_{2} - x_{1}).$$

$$(m_{1} + m_{2})l_{1}^{2}\theta_{1}'' + m_{2}l_{1}l_{2}\theta_{2}''\cos(\theta_{1} - \theta_{2}) + m_{2}l_{1}l_{2}(\theta_{2}')^{2}\sin(\theta_{1} - \theta_{2}) + (m_{1} + m_{2})l_{1}g\sin\theta_{1} = 0$$

$$m_{2}l_{2}^{2}\theta_{2}'' + m_{2}l_{1}l_{2}\theta_{1}''\cos(\theta_{1} - \theta_{2}) - m_{2}l_{1}l_{2}(\theta_{1}')^{2}\sin(\theta_{1} - \theta_{2}) + m_{2}l_{2}g\sin\theta_{2} = 0.$$

$$m_{2}l_{2}^{2}\theta_{2}'' + m_{2}l_{1}l_{2}\theta_{1}''\cos(\theta_{1} - \theta_{2}) - m_{2}l_{1}l_{2}(\theta_{1}')^{2}\sin(\theta_{1} - \theta_{2}) + m_{2}l_{2}g\sin\theta_{2} = 0.$$

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$(m_1 + m_2) l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2) l_1 g \theta_1 = 0$$

$$m_2 x_2'' = -k_2 (x_2 - x_1).$$

$$m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 l_2 g \theta_2 = 0.$$

### **Summary - 7.6: Systems of Linear Equations**

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 $L\frac{di_1}{dt} + Ri_2 = E(t)$   $i_1 L$ 

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 x_2'' = -k_2 (x_2 - x_1).$$

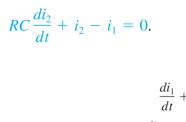
Solve

$$(x_1)$$
.

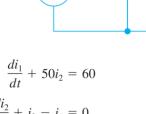
$$k_2(x_2 - x_1)$$
.  
 $x_1'' + 10x_1 - 4x_2 = 0$   
 $-4x_1 + x_2'' + 4x_2 = 0$ 

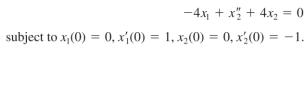
$$x_1 - 4x_2 = 0$$

$$x_1 + x_2'' + 4x_2 = 0$$



$$\frac{di_1}{dt} + 50(10^{-4}) \frac{di_2}{dt} +$$





$$50(10^{-4})\frac{di_2}{dt} + i_2 - i_1 = 0$$

$$s^{2}X_{1}(s) - sx_{1}(0) - x'_{1}(0) + 10X_{1}(s) - 4X_{2}(s) = 0$$

$$-4X_{1}(s) + s^{2}X_{2}(s) - sx_{2}(0) - x'_{2}(0) + 4X_{2}(s) = 0,$$

$$(s) = \frac{s^{2}}{s^{2}} - \frac{1}{s^{2}} + \frac{6}{5}$$

$$sI_1(s) + 50I_2(s) = \frac{60}{s}$$
  
-200 $I_1(s) + (s + 200)I_2(s) = 0$ ,

$$50I_2(s) = \frac{60}{s}$$
  
0)I\_2(s) = 0,

 $i_1(0) = 0, i_2(0) = 0.$ 

$$X_{1}(s) = \frac{s^{2}}{(s^{2} + 2)(s^{2} + 12)} = -\frac{1/5}{s^{2} + 2} + \frac{6/5}{s^{2} + 12},$$

$$x_{1}(t) = -\frac{1}{5\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{2}}{s^{2} + 2} \right\} + \frac{6}{5\sqrt{12}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{12}}{s^{2} + 12} \right\}$$

$$= -\frac{\sqrt{2}}{10} \sin \sqrt{2}t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}t.$$

$$I_1(s) = \frac{1}{s(s+100)^2} = \frac{7}{s} - \frac{7}{s+100} - \frac{7}{(s+100)^2}$$

$$I_2(s) = \frac{12,000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}$$

$$I_1(s) = \frac{60s + 12,000}{s(s + 100)^2} = \frac{6/5}{s} - \frac{6/5}{s + 100} - \frac{60}{(s + 100)^2}$$

$$I_1(s) = \frac{12,000}{s(s + 100)^2} - \frac{6/5}{s} - \frac{6/5}{s + 100} - \frac{120}{s(s + 100)^2}$$

**(b)** plot of  $x_2(t)$  vs. t

0.2

(a) equilibrium (b) motion (c) forces

$$X_{2}(s) = -\frac{s^{2} + 6}{(s^{2} + 2)(s^{2} + 12)} = -\frac{2/5}{s^{2} + 2} - \frac{3/5}{s^{2} + 12}$$

$$x_{2}(t) = -\frac{2}{5\sqrt{2}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{2}}{s^{2} + 2} \right\} - \frac{3}{5\sqrt{12}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{12}}{s^{2} + 12} \right\}$$

$$= -\frac{\sqrt{2}}{5} \sin \sqrt{2}t - \frac{\sqrt{3}}{10} \sin 2\sqrt{3}t.$$

$$i_1(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}$$

$$i_2(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}$$

$$0te^{-100t}$$