Fall 2019

微分方程
Differential Equations

Unit 07.1
Definition of Laplace Transform

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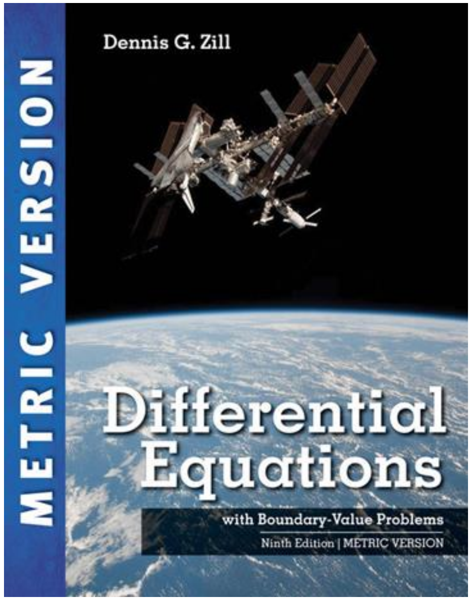
$$\mathcal{L}\left\{f(t)\right\} = F(s) \triangleq \int_0^{\infty} e^{-st} f(t) \; dt$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

7.1: DEFINTION OF THE LAPLACE TRANSFORM

- electrical

$$L \frac{dq^2(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = E(t)$$

- mechamical

$$m \frac{dx^2(t)}{dt^2} + \beta \frac{dx(t)}{dt} + k\, x(t) = f(t)$$

$$\Rightarrow a \frac{dx^2(t)}{dt^2} + b \frac{dx(t)}{dt} + c\, x(t) = g(t)$$

$$\Rightarrow a\, m^2 +$$

7.1: Transform / Operator

$f(t) \longrightarrow$

operator

$f(t) \longrightarrow$

operator

$f(t) \longrightarrow$

operator

$f(t) \longrightarrow$

operator

7.1: Defintion 7.1.1: Laplace Transform

- f: a function defined for $t \geq 0$

  or, f: $[0, \infty) \to \mathbb{R}$

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^{\infty} \quad f(t) \; dt \quad =$$

is said to be the of $f(t)$,

provided that the integral converges.

Pierre Simon Marquis de Laplace (1749-1827)

7.1: Example 1:

$$f(t) = 1$$

$$\Rightarrow \mathcal{L}\left\{ f(t) \right\} = \mathcal{L}\left\{ 1 \right\} =$$

7.1: Example 2:

$$f(t) = t$$

$$\Rightarrow \mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{t\right\} =$$

7.1: Example 3:

$$f(t) = e^{-3t}$$

$$\Rightarrow \mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{e^{-3t}\right\} =$$

7.1: Example 4:

$$f(t) = \sin 2t$$

$$\Rightarrow \mathcal{L}\left\{ f(t) \right\} = \mathcal{L}\left\{ \sin 2t \right\} =$$

7.1: Theorem 7.1.1: Some Laplace Transforms

$$(a)\quad \mathcal{L}\left\{1\right\} = \frac{1}{s}$$

$$(b)\quad \mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \cdots$$

$$(c)\quad \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$(d)\quad \mathcal{L}\left\{\sin kt\right\} = \frac{k}{s^2+k^2} \qquad (e)\quad \mathcal{L}\left\{\cos kt\right\} = \frac{s}{s^2+k^2}$$

$$(f)\quad \mathcal{L}\left\{\sinh kt\right\} = \frac{k}{s^2-k^2} \qquad (g)\quad \mathcal{L}\left\{\cosh kt\right\} = \frac{s}{s^2-k^2}$$

For others, see the Table of Laplace Transforms

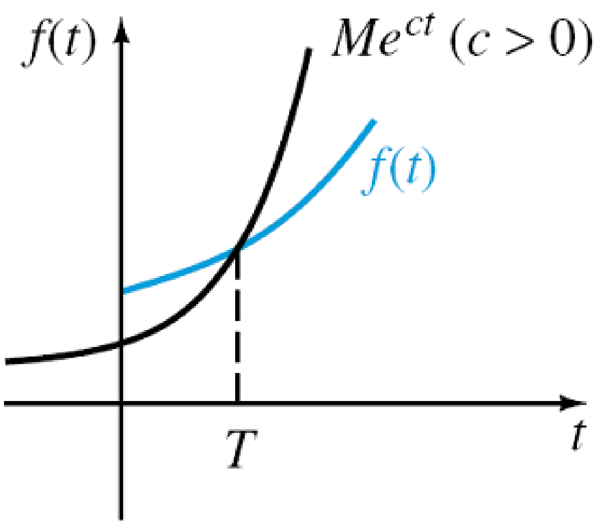
- Laplace transform is a linear transform

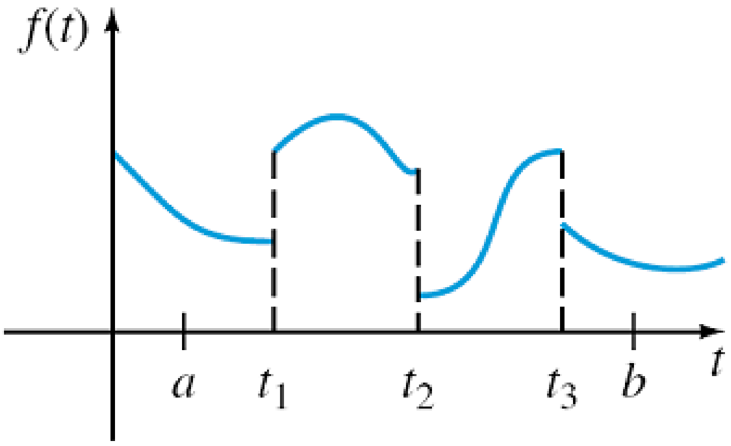
7.1: Theorem 7.1.2: Sufficient Conditions for Existence

(1) $f(t)$ is on $[0, \infty)$

(2) $f(t)$ is of for $t > T$

$$\Rightarrow \mathcal{L}\left\{ f(t) \right\} \text{ exists for } \mathcal{R}e\{s\} > c$$





$\Rightarrow \quad |f(t)| \leq M\, e^{c\,t}, \quad \forall t > T$

$\Rightarrow \quad f(t)$ is continuous on $[t_k, t_{k+1}]$

7.1: Proof of Theorem 7.1.2

7.1: Theorem 7.1.3: Behavior of F(s) as s -> infinity

(1) $f(t)$ is on $(0, \infty)$

(2) $f(t)$ is of

$$F(s) = \mathcal{L}\left\{ f(t) \right\}$$

$$\Rightarrow \lim_{s \to \infty} F(s) =$$

- Proof:

7.1: Theorem 7.1.3:

Summary - 7.1: Laplace Transform

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$(a)\quad \mathcal{L}\left\{1\right\} = \frac{1}{s}$$

$$(b)\quad \mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \cdots$$

$$(c)\quad \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$(d)\quad \mathcal{L}\left\{\sin kt\right\} = \frac{k}{s^2+k^2} \qquad (e)\quad \mathcal{L}\left\{\cos kt\right\} = \frac{s}{s^2+k^2}$$

$$(f)\quad \mathcal{L}\left\{\sinh kt\right\} = \frac{k}{s^2-k^2} \qquad (g)\quad \mathcal{L}\left\{\cosh kt\right\} = \frac{s}{s^2-k^2}$$