

Fall 2019

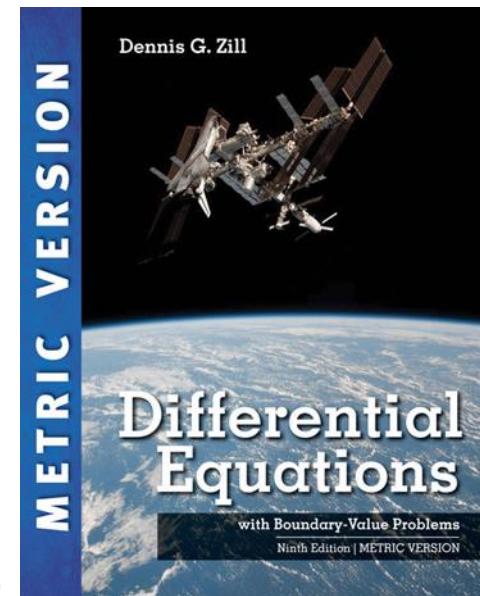
# 微分方程 Differential Equations

## Unit 07.0 Introduction to Laplace Transform

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NTU-EE

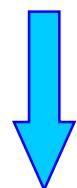
Sep19 – Jan20



Figures and images used in these lecture notes are adopted from  
**Differential Equations with Boundary-Value Problems**, 9th Ed., D.G. Zill, 2018 (Metric Version)

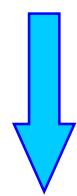
## 7.0: Transformation Concept

$$y'' + 7y' + 10y = e^{-3t} \quad \xrightarrow{\text{red arrow}} \quad s^2 Y(s) + 7sY(s) + 10Y(s) = F(s)$$



$$y(t) = e^{mt} \quad (4.3)$$

$$m^2 + 7m + 10 = 0$$



$$m = -2, -5$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-5t}$$

$$\begin{aligned} Y(s) &= \frac{F(s)}{s^2 + 7s + 10} \\ &= \frac{F(s)}{(s + 2)(s + 5)} \\ s &= -2, -5 \\ &= \frac{A(s)}{(s + 2)} + \frac{B(s)}{(s + 5)} \end{aligned}$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-5t}$$

## 7.0: Transformation Pairs

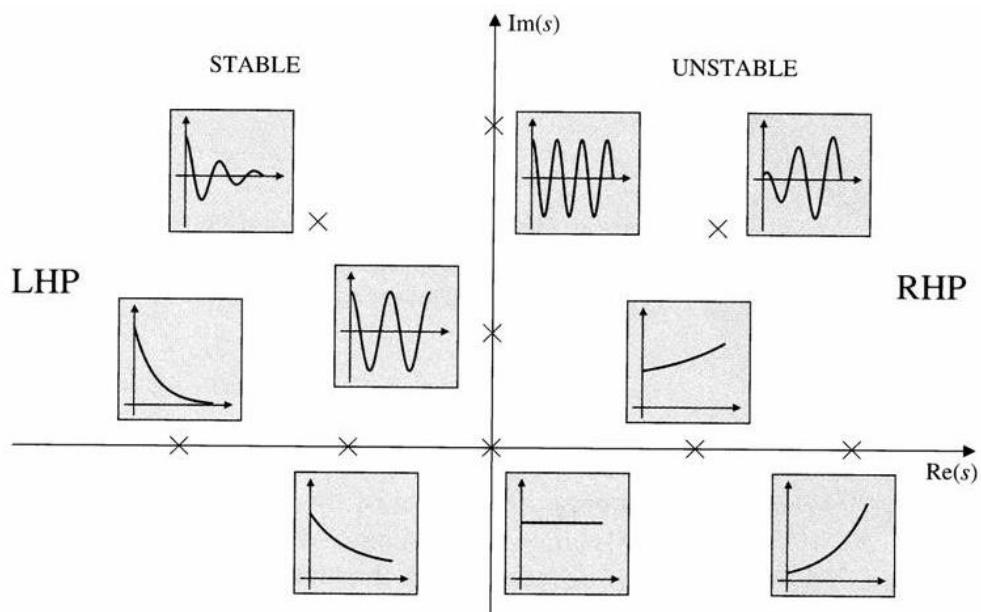
$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. $t$	$\frac{1}{s^2}$
3. $t^n$	$\frac{n!}{s^{n+1}}, \quad n \text{ a positive integer}$
7. $\sin kt$	$\frac{k}{s^2 + k^2}$
8. $\cos kt$	$\frac{s}{s^2 + k^2}$
11. $e^{at}$	$\frac{1}{s - a}$
16. $te^{at}$	$\frac{1}{(s - a)^2}$
18. $e^{at} \sin kt$	$\frac{k}{(s - a)^2 + k^2}$
19. $e^{at} \cos kt$	$\frac{s - a}{(s - a)^2 + k^2}$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

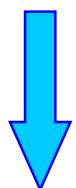
$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0),$$



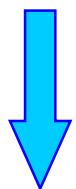
## 7.0: Transformation Concept

$$y'' + 7y' + 10y = e^{-3t} \quad \xrightarrow{\hspace{1cm}} \quad s^2 Y(s) + 7sY(s) + 10Y(s) = F(s)$$



$$y(t) = e^{mt} \quad (4.3)$$

$$m^2 + 7m + 10 = 0$$



$$m = -2, -5$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-5t}$$

$$y'_1 = y_2$$

$$y'_2 = -10y_1 - 7y_2 + e^{-3t}$$



$$\begin{aligned} Y(s) &= \frac{F(s)}{s^2 + 7s + 10} \\ &= \frac{F(s)}{(s + 2)(s + 5)} \\ s &= -2, -5 \\ &= \frac{A(s)}{(s + 2)} + \frac{B(s)}{(s + 5)} \end{aligned}$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-5t}$$



$$\text{eig} \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} = -2, -5$$

$$\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

$$y' = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} y + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

- 7.1: Definition of Laplace Transform
- 7.2: Inverse Transforms and Transforms of Derivatives
  - 7.2.1: Inverse Transforms
  - 7.2.2: Transforms of Derivatives
- 7.3: Operational Properties I
  - 7.3.1: Translation on the s-Axis
  - 7.3.2: Translation on the t-Axis
- 7.4: Operational Properties II
  - 7.4.1: Derivatives of a Transform
  - 7.4.2: Transforms of Integrals
  - 7.4.3: Transform of a Periodic Function
- 7.5: The Dirac Delta Function
- 7.6: Systems of Linear Differential Equations

$$\mathcal{L} \left\{ f(t) \right\} \triangleq \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$\mathcal{L} \left\{ f(t) \right\} = \frac{1}{1 - e^{sT}} \int_0^T e^{-st} f(t) dt$$

$$\mathcal{L} \left\{ e^{at} f(t) \right\} = F(s - a)$$

$$\mathcal{L} \left\{ f(t-a) \mathcal{U}(t-a) \right\} = e^{-as} F(s)$$

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau = h(t)$$

$$\int_a^b u v' dx = u v \Big|_a^b - \int_a^b u' v dx$$

$$f(t) = g(t) + \int_0^t f(\tau) h(t-\tau) d\tau$$

## Sample Problems

$$\frac{s^3 + 1}{s^3 - 1}$$

$$\frac{1}{s(s+1)(s^2+s+1)}$$

$$\frac{s^2 e^{-2s}}{(s+1)(s+2)}$$

$$\ln \frac{s+1}{s+2}$$

$$\sinh(t) \cos^2(t)$$

$$\int_0^t (t-\tau) \cos(t-\tau) \delta(\tau-1) d\tau$$

$$\int_0^\infty (t^5 + t^4) e^{-2t} dt$$