Fall 2019

微分方程
Differential Equations

Unit 07.0
Introduction to Laplace Transform

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NTU-EE

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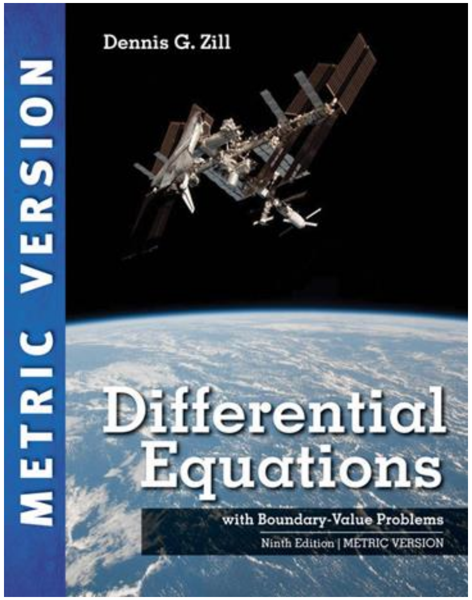


Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

7.0: Transformation Concept

$$y'' + 7y' + 10y = e^{-3t} \quad \Longrightarrow \quad s^2Y(s) + 7sY(s) + 10Y(s) = F(s)$$

$$y(t) = e^{mt} \text{ (4.3)}$$

$$m^2 + 7m + 10 = 0$$

$$m = -2, -5$$

$$y(t) = c_1e^{-2t} + c_2e^{-5t}$$

$$Y(s) = \frac{F(s)}{s^2 + 7s + 10}$$

$$= \frac{F(s)}{(s+2)(s+5)}$$

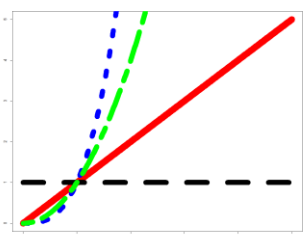
$$s = -2, -5$$

$$= \frac{A(s)}{(s+2)} + \frac{B(s)}{(s+5)}$$

$$y(t) = c_1e^{-2t} + c_2e^{-5t}$$

7.0: Transformation Pairs

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. t	$\frac{1}{s^2}$
3. t^n	$\frac{n!}{s^{n+1}}$, n a positive integer
7. $\sin kt$	$\frac{k}{s^2 + k^2}$
8. $\cos kt$	$\frac{s}{s^2 + k^2}$
11. e^{at}	$\frac{1}{s - a}$
16. te^{at}	$\frac{1}{(s - a)^2}$
18. $e^{at} \sin kt$	$\frac{k}{(s - a)^2 + k^2}$
19. $e^{at} \cos kt$	$\frac{s - a}{(s - a)^2 + k^2}$

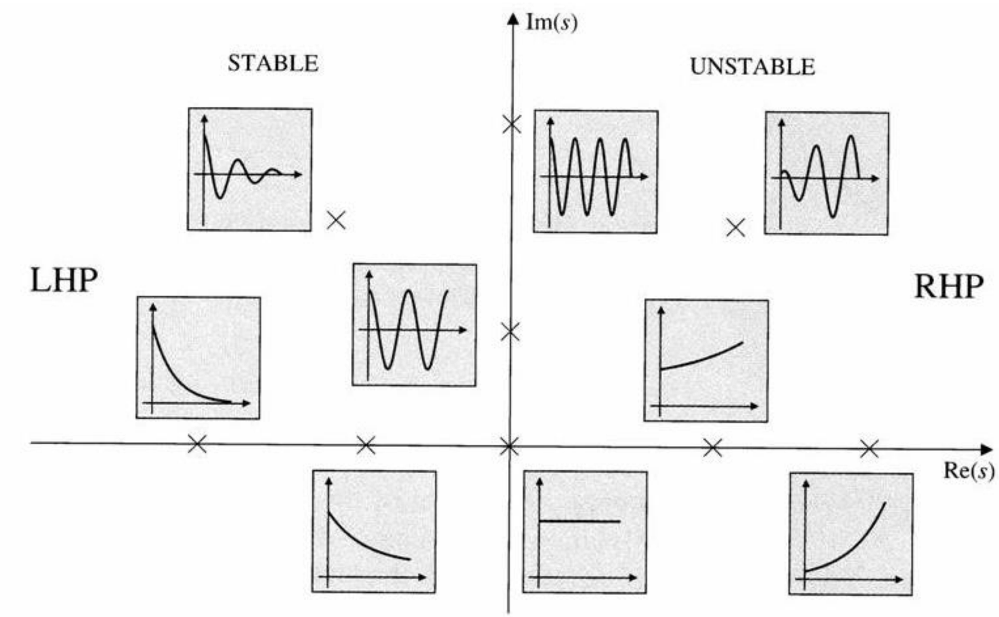


$$\mathscr{L}\{f'(t)\} = sF(s) - f(0).$$

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

$$\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$



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$$y_1' = y_2$$

$$y_2' = -10y_1 - 7y_2 + e^{-3t}$$

$$\text{eig} \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} = -2, -5$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

Outline

Important Formulae

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\mathcal{L}\left\{f(t)\right\} = \frac{1}{1-e^{sT}} \int_0^T e^{-st} f(t)\, dt$$

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

$$\mathcal{L}\left\{f(t-a)\,\mathcal{U}(t-a)\right\} = e^{-as} F(s)$$

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = h(t)$$

$$\int_a^b u\, v'\, dx = u\, v\Big|_a^b - \int_a^b u'\, v\, dx$$

$$f(t) = g(t) + \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Sample Problems

$$\frac{s^3+1}{s^3-1}$$

$$\frac{1}{s(s+1)(s^2+s+1)}$$

$$\frac{s^2 e^{-2s}}{(s+1)(s+2)}$$

$$\ln\frac{s+1}{s+2}$$

$$\sinh(t)\cos^2(t)$$

$$\int_0^t (t-\tau)\cos(t-\tau)\,\delta(\tau-1)\,d\tau$$

$$\int_0^\infty (t^5+t^4)e^{-2t}\,dt$$