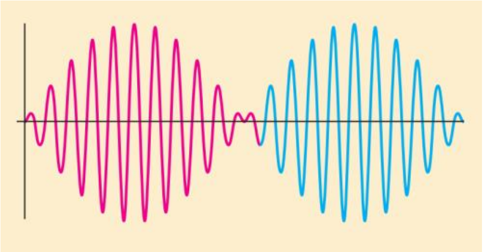
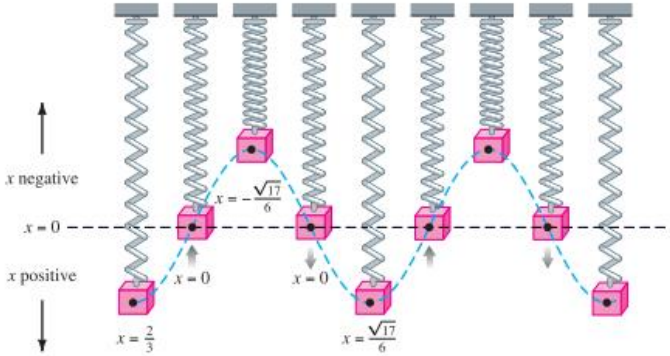


Fall 2019

微分方程
Differential Equations

Unit 05.1
Linear Models: Initial-Value Problems



Feng-Li Lian

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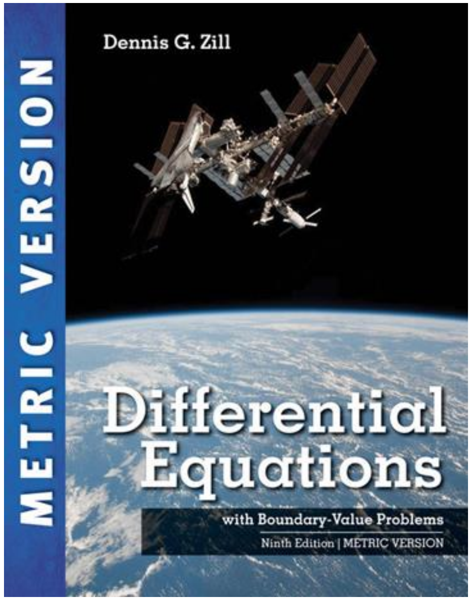
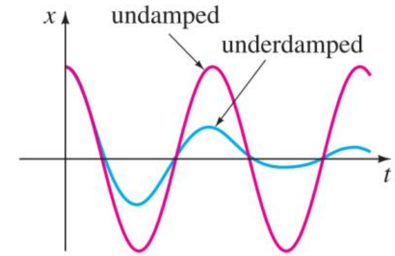
Sep19 – Jan20

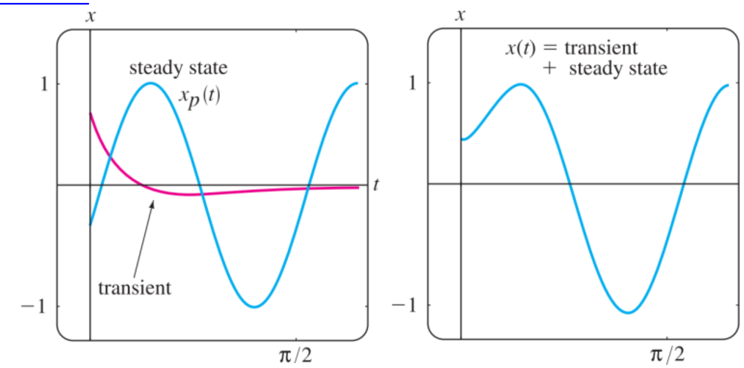


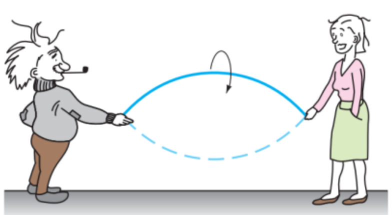
Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

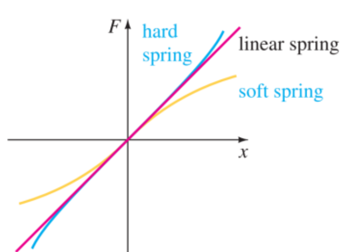
Outline

- **5.1: Linear Models: Initial-Value Problems**
  - **5.1.1: Spring/Mass Systems: Free Undamped Motion**
  - **5.1.2: Spring/Mass Systems: Free Damped Motion**
  - **5.1.3: Spring/Mass Systems: Driven Motion**
  - **5.1.4: Series Circuit Analogue**
- 5.2: Linear Models: Boundary-Value Problems
- 5.3: Nonlinear Models









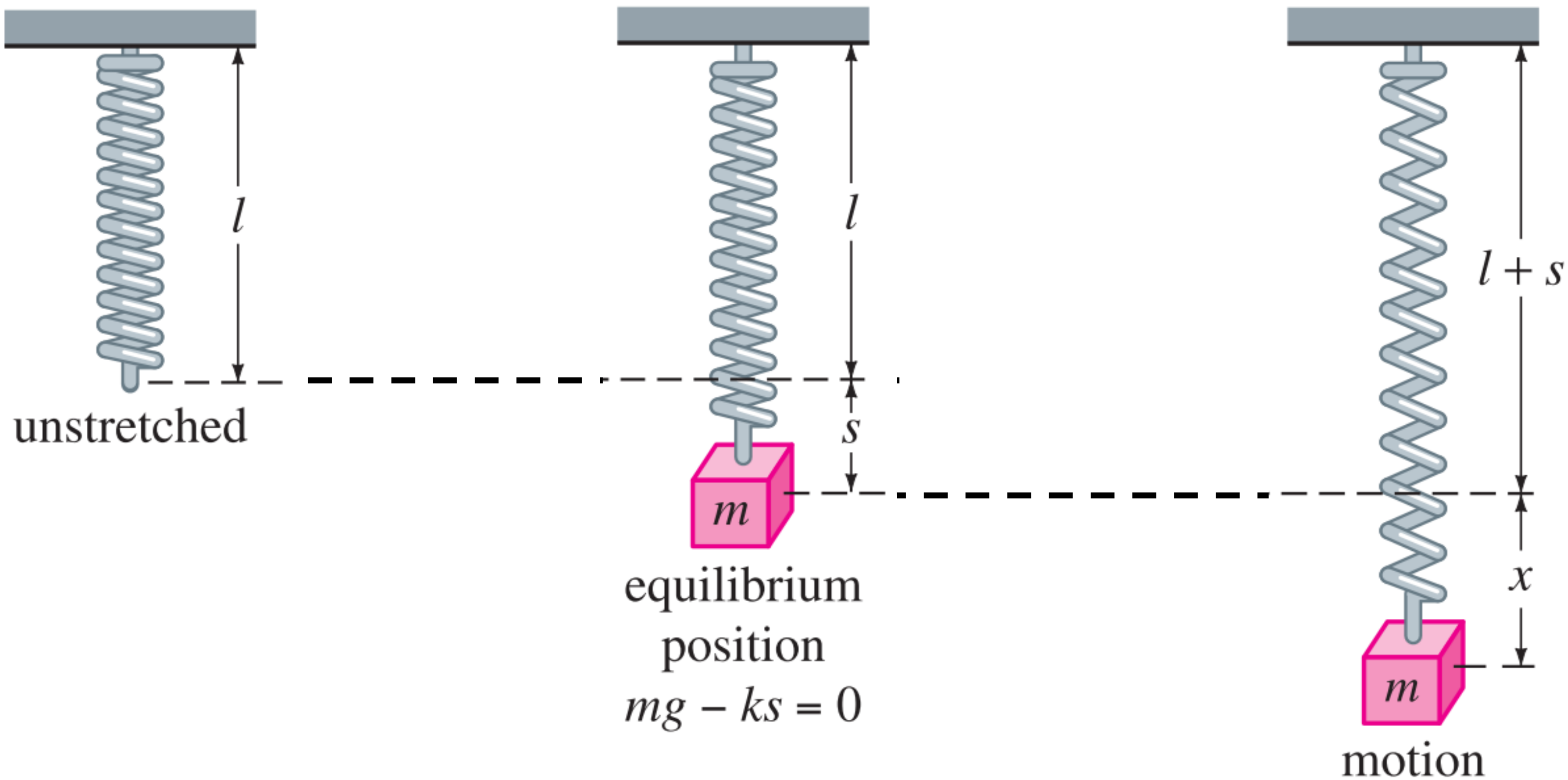
$$m\,\frac{dx^2(t)}{dt^2} + k\,x(t) = 0$$

$$m\,\frac{dx^2(t)}{dt^2} + \beta\,\frac{dx(t)}{dt} + k\,x(t) = 0$$

$$m\,\frac{dx^2(t)}{dt^2} + \beta\,\frac{dx(t)}{dt} + k\,x(t) = f(t)$$

$$\Rightarrow L\,\frac{dq^2(t)}{dt^2} + R\,\frac{dq(t)}{dt} + \frac{1}{C}\,q(t) = E(t)$$

- free undamped motion:
  - → free: no external force
  - → undamped: no friction



- Hooke's law:
- Newton's 2nd law:

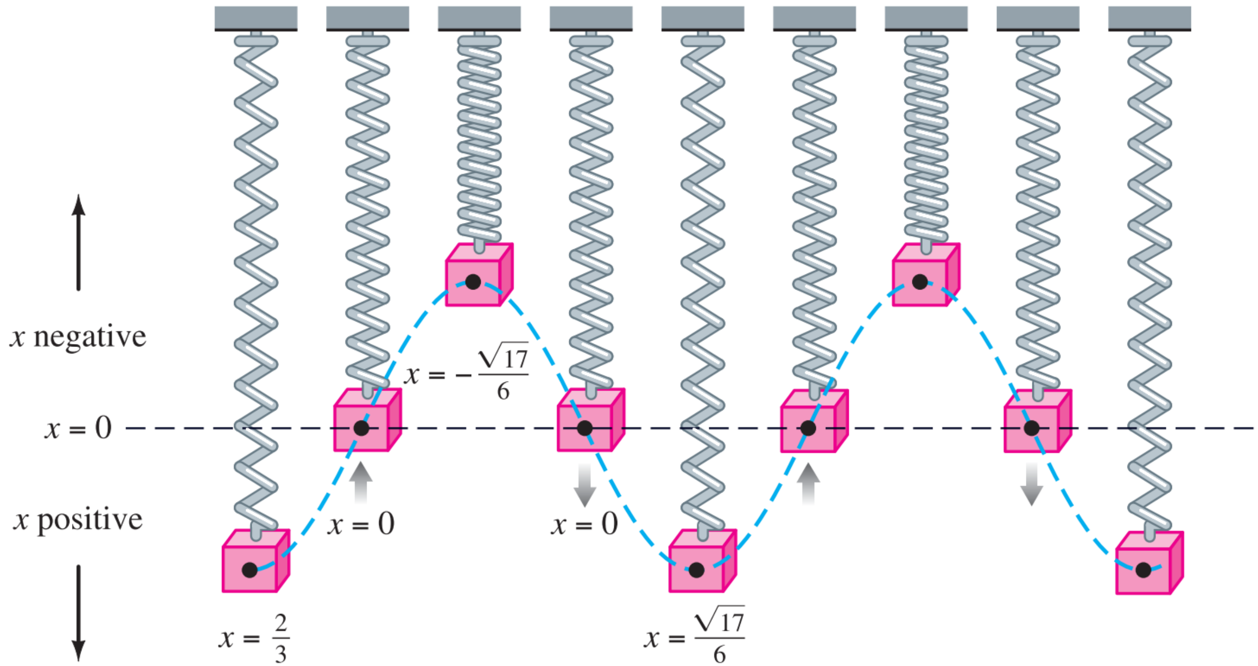
5.1.1:

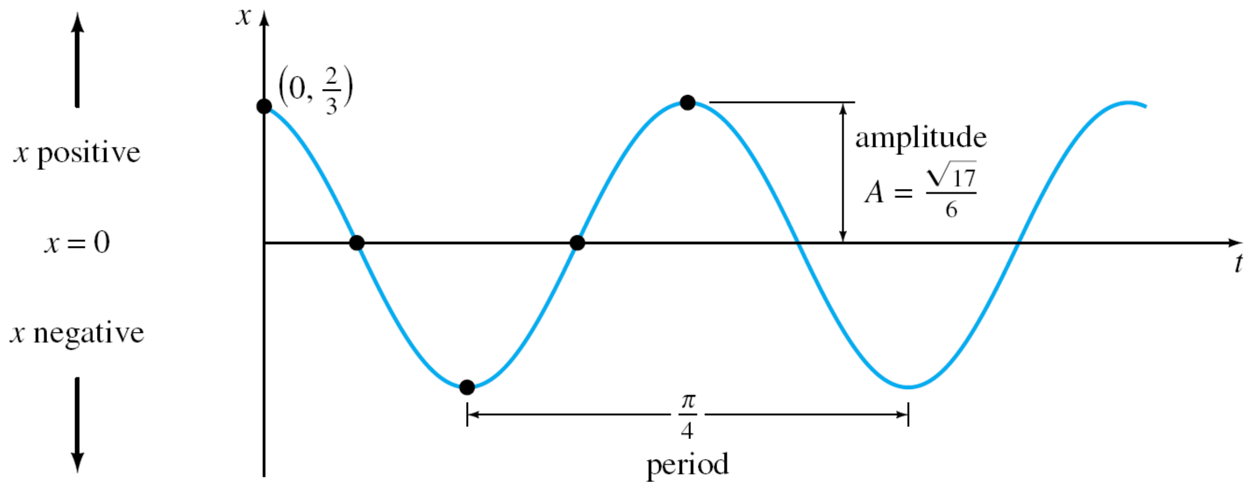
$$\Rightarrow \; m\frac{d^2x}{dt^2} \; =$$

$$\Rightarrow \; m\frac{d^2x}{dt^2} \; +$$

$$\Rightarrow \; \frac{d^2x}{dt^2} \; +$$

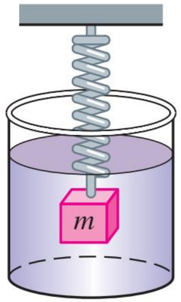
$$\Rightarrow \; x(t) \; =$$


x negative
x = 0
x positive
$x = -\frac{\sqrt{17}}{6}$
x = 0
x = 0
$x = \frac{2}{3}$
$x = \frac{\sqrt{17}}{6}$

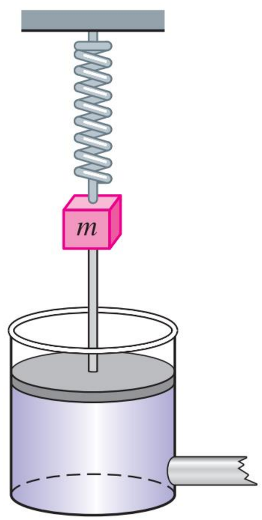

x
$(0, \frac{2}{3})$
x positive
x = 0
x negative
amplitude
$A = \frac{\sqrt{17}}{6}$
t
$\frac{\pi}{4}$
period

(b)

- free damped motion: → damped: friction



$$\Rightarrow m\frac{d^2x}{dt^2} =$$



$$\Rightarrow \frac{d^2x}{dt^2} +$$

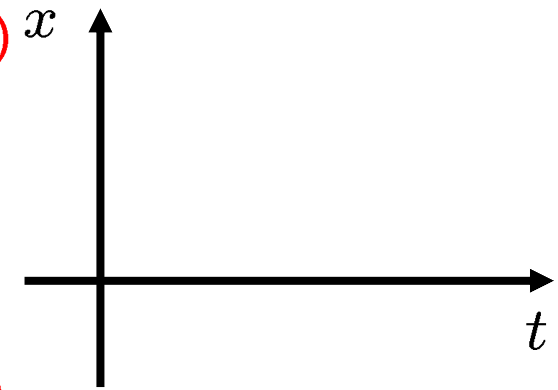
$$\Rightarrow m^2 +$$

$$\Rightarrow m_{1,2} =$$

5.1.2:

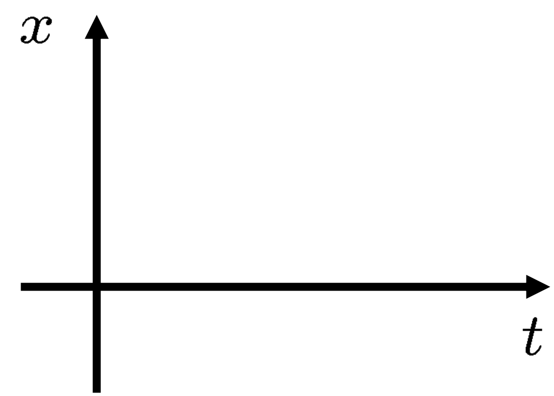
(1) $\lambda^2 - w^2 \quad 0 \quad (\qquad)$

$\Rightarrow x(t) =$



(2) $\lambda^2 - w^2 \quad 0 \quad (\qquad)$

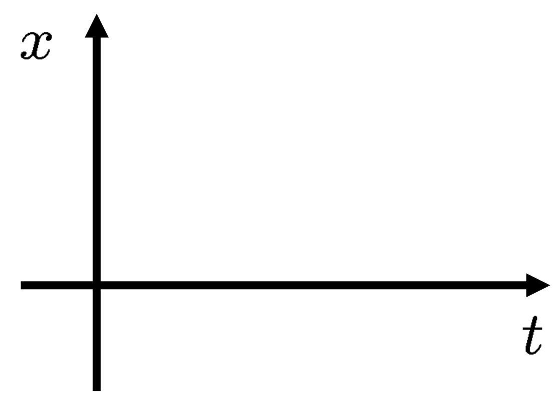
$\Rightarrow x(t) =$



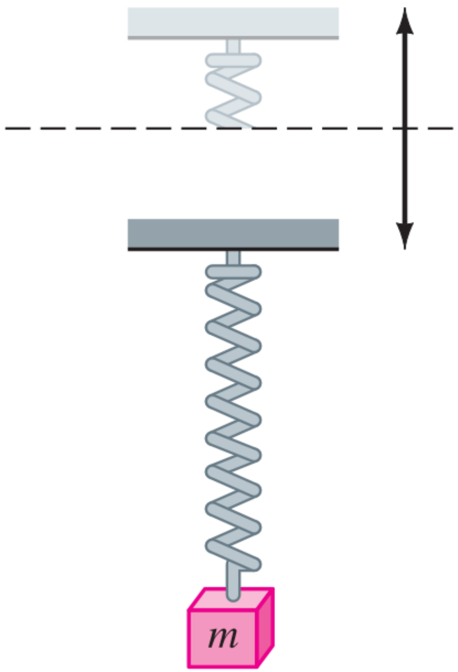
(3) $\lambda^2 - w^2 \quad 0 \quad (\qquad)$

$\Rightarrow x(t) =$

or $x(t) =$



- driven motion: $\longrightarrow$ with external force



$$\Rightarrow \quad m\frac{d^2x}{dt^2} =$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} +$$

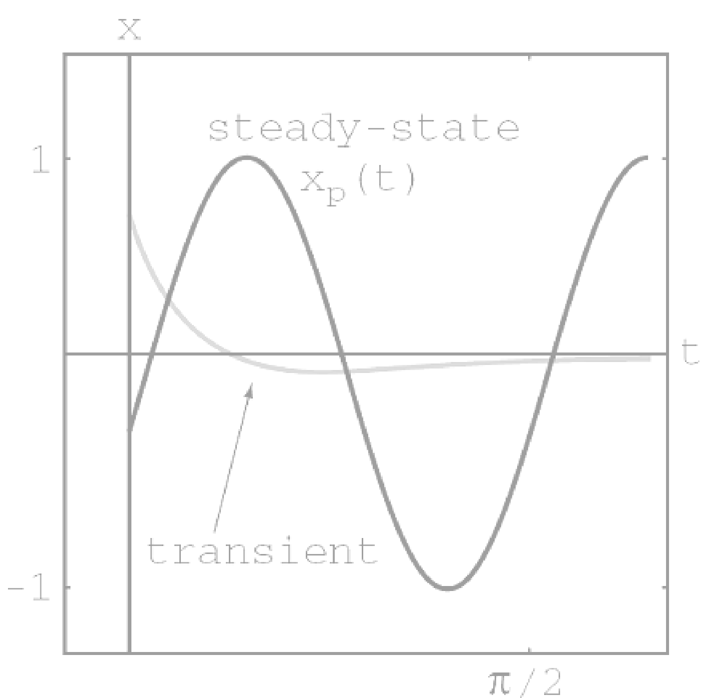
- consider underdamped case:

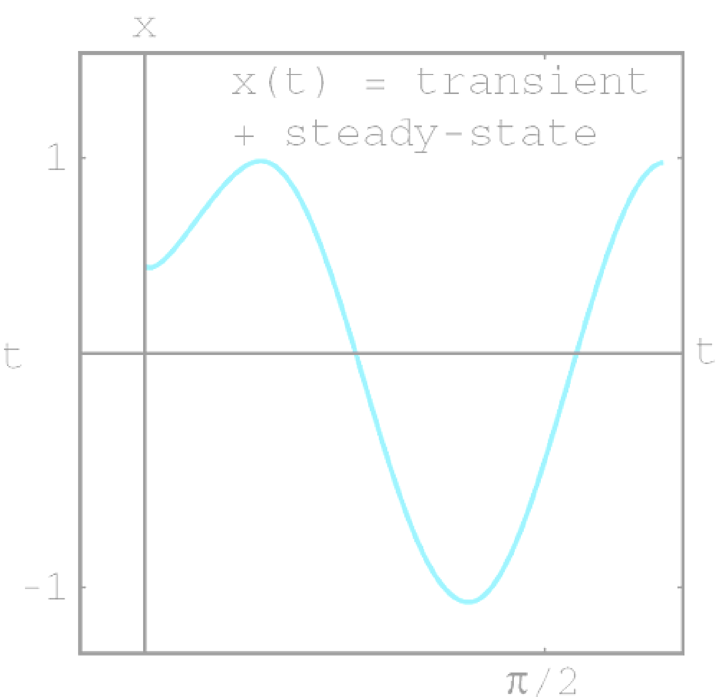
- IF $F(t) = F_0 \sin rt$

$\Rightarrow x_c(t) =$

$\Rightarrow x_p(t) =$

$\Rightarrow x(t) =$





- Resonance

$\Rightarrow \lambda =$

$\Rightarrow \dfrac{d^2x}{dt^2} +$

$\Rightarrow x_c(t) =$

$\Rightarrow x_p(t) =$

$\Rightarrow x'_p(t) =$

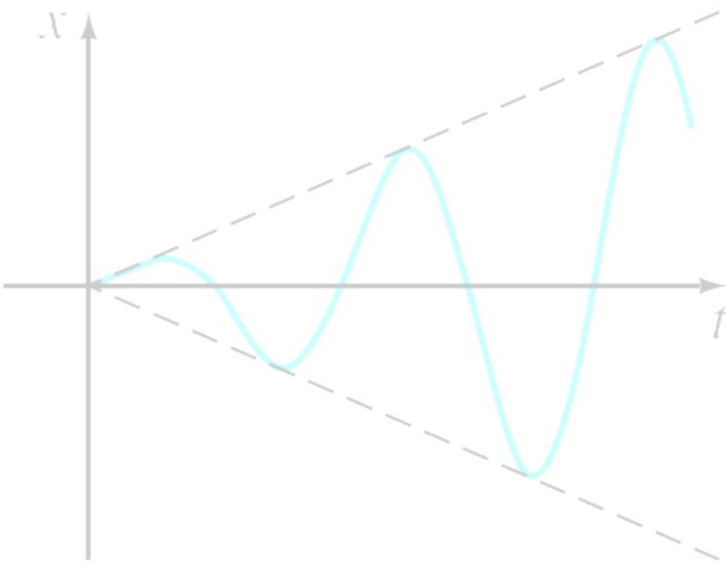
$\Rightarrow x''_p(t) =$

$\Rightarrow \quad x_p'' + w^2 x_p =$

$\Rightarrow$

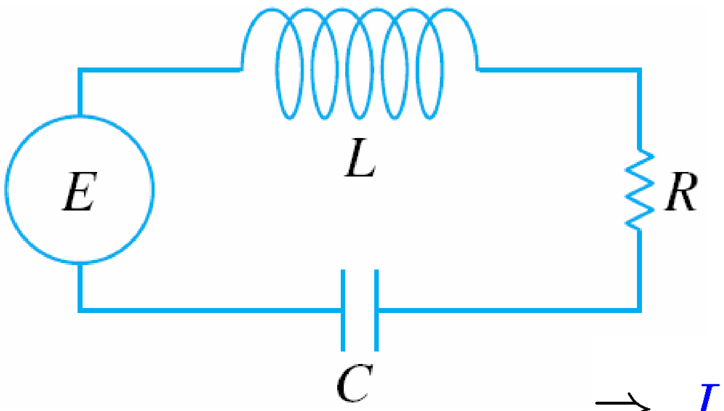
$\Rightarrow \quad x_p(t) =$

$\Rightarrow \quad x(t) =$



$\Rightarrow \quad x(t) =$

5.1.4: Series Circuit Analogue



$$\Rightarrow \quad L\,\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int i(t)dt = E(t)$$

$$\Rightarrow \quad i(t) = \frac{dq(t)}{dt} \qquad \Rightarrow \quad L\,\frac{dq^2(t)}{dt^2} + R\,\frac{dq(t)}{dt} + \frac{1}{C}\,q(t) = E(t)$$

$$\text{OR} \quad L\,\frac{di^2(t)}{dt^2} + R\,\frac{di(t)}{dt} + \frac{1}{C}\,i(t) = \frac{dE(t)}{dt}$$

$$m\,\frac{dx^2(t)}{dt^2} + \beta\,\frac{dx(t)}{dt} + k\,x(t) = f(t)$$

- IF $E(t) = 0$

$\Rightarrow$ electrical vibrations of the circuit

- auxiliary eqn:

- IF $R \neq 0$, the circuit is

damped if $R^2 - \frac{4L}{C}$ 0

damped if $R^2 - \frac{4L}{C}$ 0

damped if $R^2 - \frac{4L}{C}$ 0

5.1.4:

- IF $E(t) = E_0 \sin rt$

$\Rightarrow$ electrical vibrations of the circuit

$$\Rightarrow L \frac{dq^2(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = E_0 \sin rt$$

$\Rightarrow q_c(t) =$

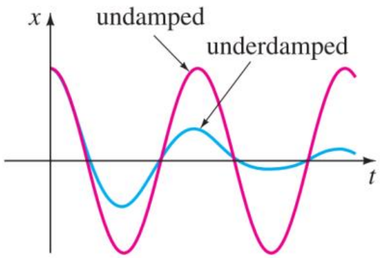
$\Rightarrow q_p(t) =$

$\Rightarrow i_p(t) =$

$$\frac{dx^2(t)}{dt^2} + 2\lambda \frac{dx(t)}{dt} + w^2\, x(t) = f(t)$$

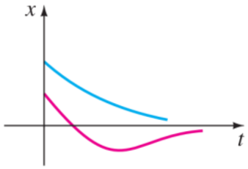
- free undamped motion:

$$\Rightarrow x(t) = c_1 \cos wt + c_2 \sin wt$$



- free damped motion:

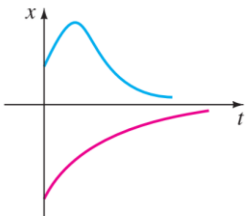
(1) $\lambda^2 - w^2 > 0$ (overdamped)



$$\Rightarrow x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2 - w^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - w^2}\,t}\right)$$

(2) $\lambda^2 - w^2 = 0$ (critical damped)

$$\Rightarrow x(t) = e^{-\lambda t}\,(c_1 + c_2 t)$$



(3) $\lambda^2 - w^2 < 0$ (underdamped)

$$\Rightarrow x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{w^2 - \lambda^2}\,t + c_2 \sin\sqrt{w^2 - \lambda^2}\,t\right)$$

$$\frac{dx^2(t)}{dt^2} + 2\lambda \frac{dx(t)}{dt} + w^2\, x(t) = f(t)$$

- driven motion:

$$\Rightarrow \; x(t) = x_c(t) + x_p(t)$$

- Resonance

$$x_c(t) = c_1 \cos wt + c_2 \sin wt$$

$$f(t) = F_1 \cos wt + F_2 \sin wt$$

$$\Rightarrow \; x_p(t) = A\, t \cos wt + B\, t \sin wt$$