

Fall 2019

# 微分方程 Differential Equations

## Unit 04.7 Cauchy-Euler Equations

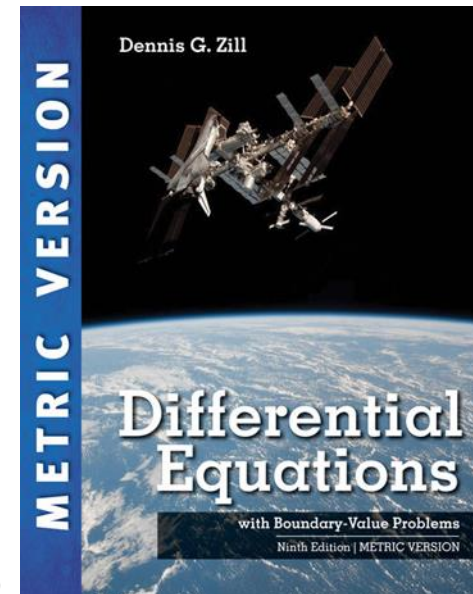
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$$ax^2y'' + bxy' + cy = 0$$

$$y(x) = x^m$$



- 4.1: Linear Differential Equations: Basic Theory
  - 4.1.1: Initial-Value and Boundary-Value Problems
  - 4.1.2: Homogeneous Equations
  - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients – Superposition Approach
- 4.5: Undetermined Coefficients – Annihilator Approach
- 4.6: Variation of Parameters
- 4.7: Cauchy-Euler Equations
- 4.8: Green's Functions
- 4.9: Solving Systems of Linear Equations by Elimination
- 4.10: Nonlinear Differential Equations

## 4.7: Cauchy-Euler Equations

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

$$a_i \in \mathbb{R}, \quad I = (0, \infty)$$

known as a **Cauchy-Euler Equation (Equidimensional Eqn)**

Augustin-Louis Cauchy (1789-1857)

Leonhard Euler (1707-1783)

- Consider a 2nd-order DE:

$$a x^2 y'' + b x y' + c y = 0$$

- Assume  $y(x) =$

$$y'(x) =$$

$$y''(x) =$$

- Case I: distinct real roots:  $m_1, m_2$

- Case II: repeated real roots:  $m_1 = m_2$

$$\begin{cases} y_1(x) = \\ y_2(x) = \end{cases}$$

$$y_2'(x) =$$

$$y_2''(x) =$$



- Case III: conjugate complex roots

$$\begin{cases} m_1 = \\ m_2 = \end{cases}$$

$$\Rightarrow x^{m_1} =$$

$$\Rightarrow x^{m_2} =$$

$$\Rightarrow y(x) =$$

$$a x^2 y'' + b x y' + c y = 0$$

• Let  $x =$

$y(x) \longrightarrow$



- auxiliary eqn:
- Case I:  $m_1, m_2$  distinct real roots
- Case II:  $m_1 = m_2$

- Case III:  $m_{1,2} = \alpha \pm i\beta$

## 4.7: Example 5

$$x^2 y'' - 3x y' + 3y = 2x^4 e^x$$

• auxiliary eqn.  $y(x) =$

$$\Rightarrow y_c =$$

$$\Rightarrow y_p =$$



## 4.7: Another Form of Cauchy-Euler Equation

$$a x^2 y'' + b x y' + c y = 0$$

$$a (x - x_0)^2 y'' + b (x - x_0) y' + c y = 0$$

• Consider a 2nd-order DE:  $a x^2 y'' + b x y' + c y = 0$

• Assume  $y(x) = x^m$

$$y'(x) = m x^{m-1}$$

$$y''(x) = m(m-1) x^{m-2}$$

$$\Rightarrow a m(m-1) + b m + c = 0$$

$$\Rightarrow a m^2 + (b-a)m + c = 0$$

• Case I: distinct real roots:  $m_1, m_2$

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}$$

• Case II: repeated real roots:  $m_1 = m_2$

$$y(x) = c_1 x^{m_1} + c_2 \ln x x^{m_1}$$

• Case III: conjugate complex roots  $m_{1,2} = a \pm ib$

$$y(x) = x^a (c_1 \cos(b \ln x) + c_2 \sin(b \ln x))$$