Fall 2019

微分方程
Differential Equations

Unit 04.6
Variation of Parameters

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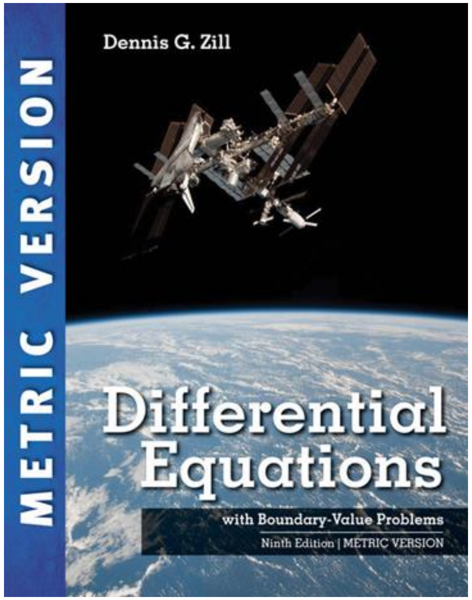
$$y_c(x) = c_1 y_1 + c_2 y_2$$

$$y_p(x) = u_1 y_1 + u_2 y_2$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- 4.1: Linear Differential Equations: Basic Theory
  - 4.1.1: Initial-Value and Boundary-Value Problems
  - 4.1.2: Homogeneous Equations
  - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients – Superposition Approach
- 4.5: Undetermined Coefficients – Annihilator Approach
- **4.6: Variation of Parameters**
- 4.7: Cauchy-Euler Equations
- 4.8: Green's Functions
- 4.9: Solving Systems of Linear Equations by Elimination
- 4.10: Nonlinear Differential Equations

- Undertermined coefficients: (4.4, 4.5)

need $\begin{cases} (1) \\ \\ (2) \end{cases}$

- If $g(x)$ does NOT belongs to these types of functions,

we CANNOT use undetermined coefficients

$\Rightarrow$ Use Variation of Parameters

Joseph Louis Lagrange (1736-1813)

- Consider a 2nd-order DE:

$$a_2(x)\, y''(x) \; + \; a_1(x)\, y'(x) \; + \; a_0(x)\, y(x) \; = \; g(x)$$

$\Rightarrow$ standard form

$$y''(x) \; + \qquad y'(x) \; + \qquad y(x) \; = \qquad g(x)$$

4.6: Example 2

$$4\,y'' + 36\,y = \csc 3x$$

$\Rightarrow$ standard form

4.6: Another Point of View

$$y'' + P(x)\, y' + Q(x)\, y = f(x)$$

- Given y_1, y_2, satisfy
- And $\left\{ y_1, y_2 \right\}$:
- How to find y_p ?

$\Rightarrow y_p =$

$$y''' + P(x)\, y'' + Q(x)\, y' + R(x)\, y = f(x)$$

- Given $y_1, \ y_2, \ y_3$:

- Assume $y_p =$

Summary - 4.6: Variation of Parameters

- Consider a 2nd-order DE:

$$y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = f(x)$$

- $\left\{ y_1(x),\ y_2(x) \right\}$: are linearly independent solutions of AHE

$$\Rightarrow \quad y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$\rightarrow \begin{cases} w_1(x) = u_1'(x) \\ w_2(x) = u_2'(x) \end{cases}$$

$$\Rightarrow \begin{cases} w_1\, y_1 + w_2\, y_2 = 0 \\ w_1\, y_1' + w_2\, y_2' = f \end{cases} \Rightarrow w_1 = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}},\ w_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$\Rightarrow \quad y_p(x) = \left(\int w_1 dx \right) y_1(x) + \left(\int w_2 dx \right) y_2(x)$$