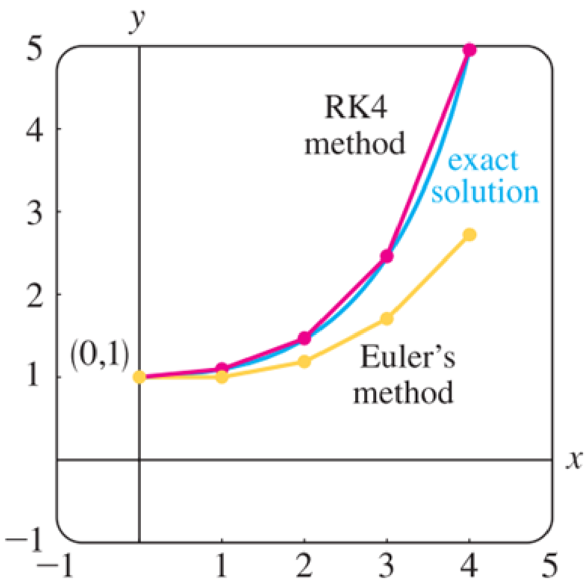
Fall 2019

微分方程
Differential Equations

Unit 02.6
A Numerical Method



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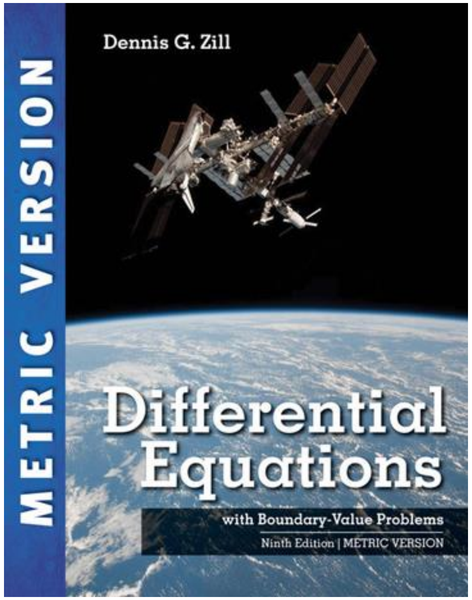
Sep19 – Jan20



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

Solutions

Solutions

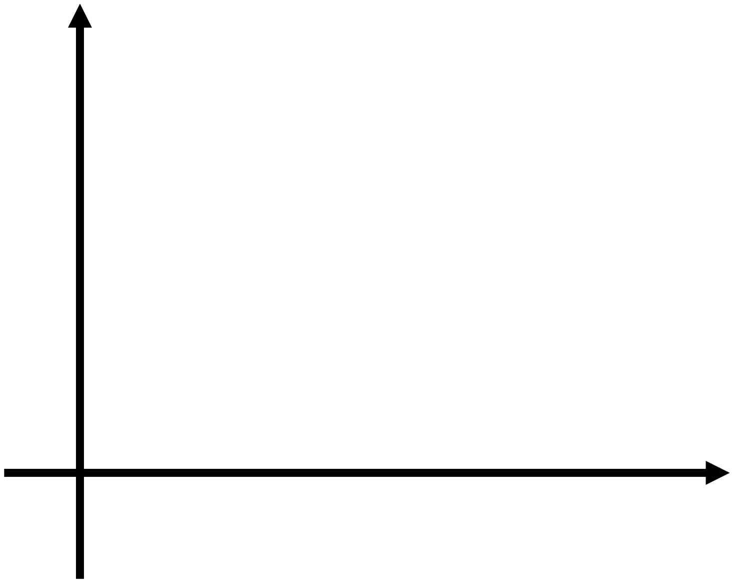
- Qualitative (§2.1)
  - Existence
  - Uniqueness
  - Asymptotes
  - Increase/Decrease
- Analytic (§2.2~§2.5)
  - Explicit functions
  - Implicit functions
  - Separable variables
  - Linear equations
  - Exact equations
  - Solutions by substitution
- Numerical (§2.6)
  - Euler's method

A Numerical Method

$$\frac{dy}{dx} = f(x, y), \qquad \text{with} \quad y(x_0) = y_0$$

- Example

$$\frac{dy}{dx} = 0.1\sqrt{y} + 0.4x^2, \qquad \text{with} \quad y(2) = 4$$



$$\frac{dy}{dx} = 0.1\sqrt{y} + 0.4x^2, \qquad \text{with} \quad y(2) = 4$$

TABLE 2.1 $h = 0.1$

x_n	y_n
2.00	4.0000
2.10	4.1800
2.20	4.3768
2.30	4.5914
2.40	4.8244
2.50	5.0768

TABLE 2.2 $h = 0.05$

x_n	y_n
2.00	4.0000
2.05	4.0900
2.10	4.1842
2.15	4.2826
2.20	4.3854
2.25	4.4927
2.30	4.6045
2.35	4.7210
2.40	4.8423
2.45	4.9686
2.50	5.0997

$\Rightarrow$ By Taylor series expansion

$$y(x) = y(x_0) + \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \cdot (x - x_0)$$

$$+ \frac{1}{2!} \left.\frac{d^2y}{dx^2}\right|_{(x_0,y_0)} \cdot (x - x_0)^2 + \text{ H.O.T.}$$

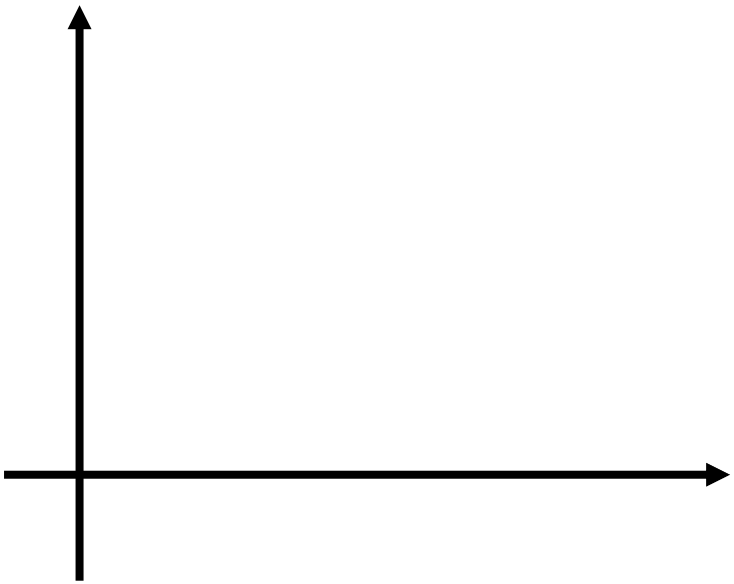
- If $x - x_0 = h$ is small,

$\Rightarrow$ H.O.T. $(x-x_0)^2$, $(x-x_0)^4$ can be neglected

$\Rightarrow$ $y(x) \approx$

A Numerical Method

- More systematically,



$$y_1 \approx y(x_1) \approx y_0 + h\, f(x_0, y_0)$$

- Let y_2 be the approximation of $y(x_2)$

  $\Rightarrow$ slope at $(x_1, y(x_1))$ is

  $\Rightarrow y_2 \approx$

- Both slope and tangent point are

- Summary

$$\frac{dy}{dx} = f(x, y), \qquad \text{with} \quad y(x_0) = y_0$$

$$\Rightarrow \quad y_{n+1} = y_n + h\, f(x_n, y_n)$$

where $x_n = x_0 + n \cdot h$ for $n = 0, 1, 2, \cdots$

h : step size

$\Rightarrow$ y_{n+1} is the approximation of $y(x_{n+1})$

A Numerical Method

- Example 2:

$$\frac{dy}{dx} = 0.2\,x\,y, \qquad \text{with} \quad y(1) = 1$$

- approximate:

- exact:

- absolute error:

- relative error:

A Numerical Method

TABLE 2.3 $h = 0.1$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.0200	1.0212	0.0012	0.12
1.20	1.0424	1.0450	0.0025	0.24
1.30	1.0675	1.0714	0.0040	0.37
1.40	1.0952	1.1008	0.0055	0.50
1.50	1.1259	1.1331	0.0073	0.64

TABLE 2.4 $h = 0.05$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.0100	1.0103	0.0003	0.03
1.10	1.0206	1.0212	0.0006	0.06
1.15	1.0318	1.0328	0.0009	0.09
1.20	1.0437	1.0450	0.0013	0.12
1.25	1.0562	1.0579	0.0016	0.16
1.30	1.0694	1.0714	0.0020	0.19
1.35	1.0833	1.0857	0.0024	0.22
1.40	1.0980	1.1008	0.0028	0.25
1.45	1.1133	1.1166	0.0032	0.29
1.50	1.1295	1.1331	0.0037	0.32

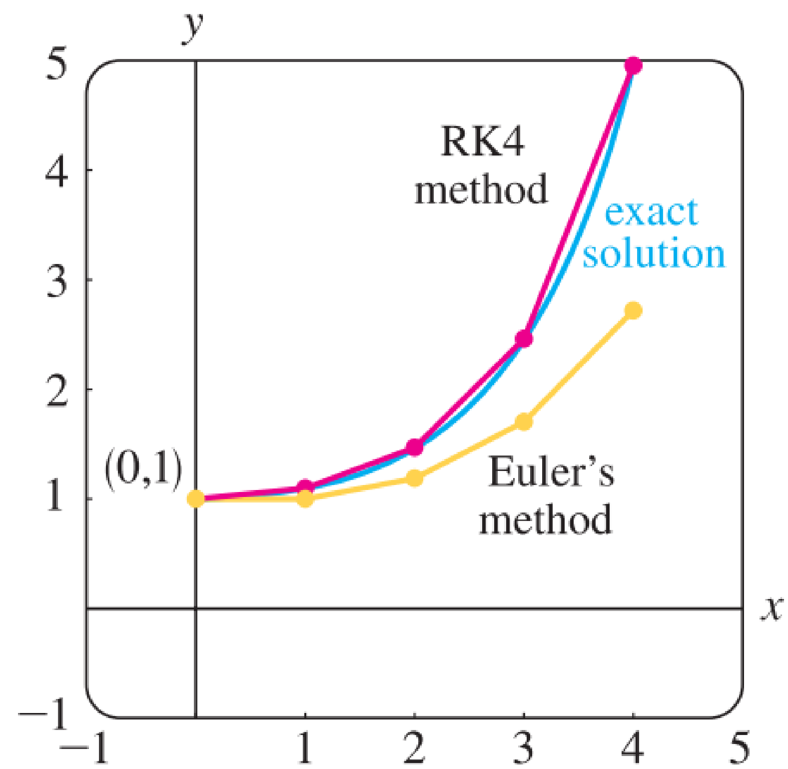


FIGURE 2.6.3 Comparison of the Runge-Kutta (RK4) and Euler methods