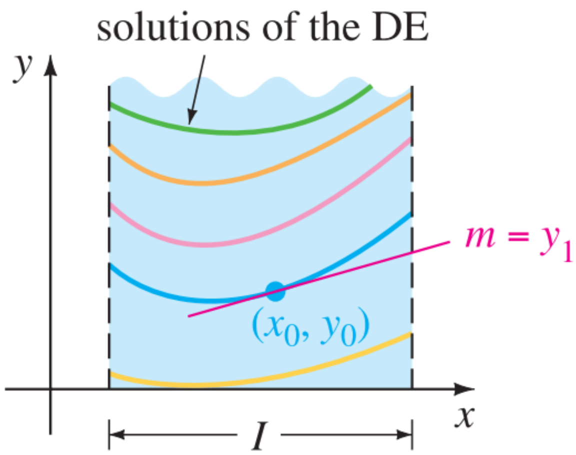
Fall 2019

微分方程
Differential Equations

Unit 01.2
Initial-Value Problems



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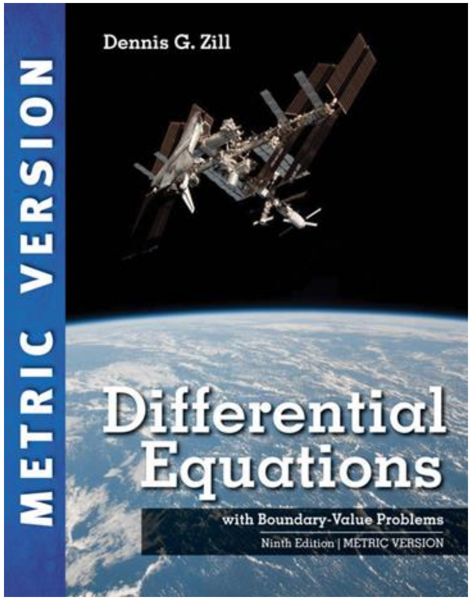
Sep19 – Jan20



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

- A solution $y(x)$ of a DE

  so that $y(x)$ also satisfies certain prescribed side conditions,

  that is, conditions that are imposed on

  the unknown function $y(x)$ and

  its derivatives at a number x_0.

- On some interval I containing x_0,

  the problem of solving an nth-order DE

  subject to n side conditions specified at x_0:

Solve: $$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \qquad \text{----------- (1)}$$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ \dots,\ y^{(n-1)}(x_0) = y_{n-1}$,

Solve: $$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \quad \text{------------(1)}$$

Subject to: $y(x_0) = y_0$, $y'(x_0) = y_1$, … , $y^{(n-1)}(x_0) = y_{n-1}$,

- is called an ***n*th-order initial-value problem (IVP)**,

  where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary constants.

- The values of $y(x)$ and its first $n-1$ derivatives at x_0,

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

  are called **initial conditions (IC).**

- Solving an *n*th-order initial-value problem such as (1)

  frequently entails first

  finding an *n*-parameter family of solutions of the DE and

  then using the initial conditions at x_0

  to determine the *n* constants in this family.

- The resulting particular solution

  is defined on some interval *I* containing the number x_0.

Geometric Interpretation

- The cases $n = 1$ and $n = 2$ in (1),

Solve: $\dfrac{dy}{dx} = f(x, y)$

------------(2)

Subject to: $y(x_0) = y_0$

Solve: $\dfrac{d^2y}{dx^2} = f(x, y, y')$

------------(3)

Subject to: $y(x_0) = y_0, \quad y'(x_0) = y_1$

- are examples of **first-** and **second-order** IVPs, respectively.

- These two problems are easy to interpret in geometric terms.

- For (2) we are seeking a solution $\boldsymbol{y(x)}$ of $\boldsymbol{y' = f(x, y)}$ on an interval $\boldsymbol{I}$ containing $\boldsymbol{x_0}$ so that its graph passes through the specified point (x_0, y_0).

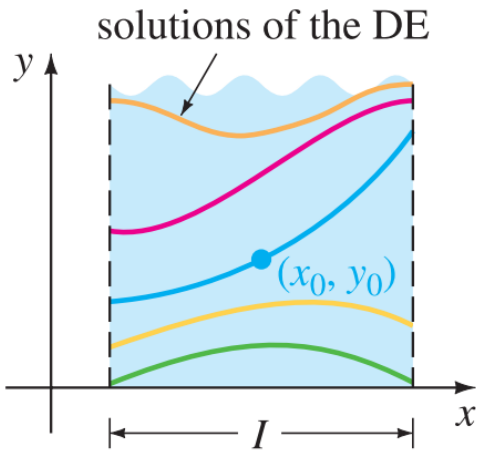


Figure 1.2.1:
Solution curve of first-order IVP

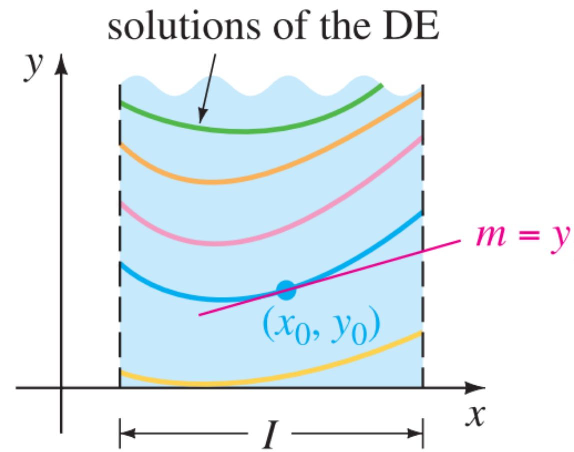


Figure 1.2.2:
Solution curve of second-order IVP

- For (3) we want to find a solution $\boldsymbol{y(x)}$ of $\boldsymbol{y'' = f\,(x, y, y')}$ on an interval $\boldsymbol{I}$ containing $\boldsymbol{x_0}$ so that its graph not only passes through $\boldsymbol{(x_0, y_0)}$ but the slope of the curve at this point is the number $\boldsymbol{y_1}$.

- Two fundamental questions arise in considering an IVP:
  - *Does a solution of the problem exist?*
  - *If a solution exists, is it unique?*

- **Existence**:
  - *Does* $\boldsymbol{dy/dx = f(x, y)}$ *possess solutions?*
  - *Do any of the solution curves pass through* $\boldsymbol{(x_0, y_0)}$?

- **Uniqueness:**
  - *When can we be certain that*

    there is precisely one solution curve

    passing through the point $\boldsymbol{(x_0, y_0)}$?

Theorem 1.2.1: Existence of a Unique Solution

- Let $\boldsymbol{R}$ be a rectangular region in the xy-plane

  defined by $a \leq x \leq b$, $c \leq y \leq d$

  that contains the point (x_0, y_0) in its interior.

- If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on $\boldsymbol{R}$,

  then there exists

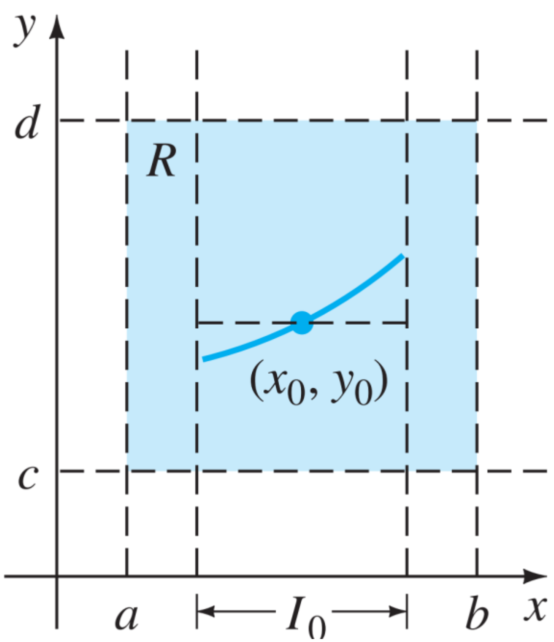
  some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$,

  contained in $[a, b]$, and

  a unique function $y(x)$,

  defined on I_0,

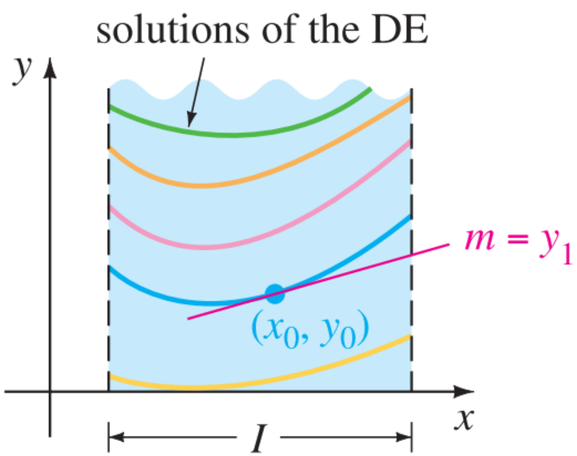
  that is a solution of the IVP (2).



$$\frac{d^2y}{dx^2} + 16y = 0 \qquad y(x) = a\cos(4x) + b\cos(4x)$$

- **IVP:** $y(0) = 2, \quad y'(0) = -3$

  $y(3) = 5, \quad y'(3) = 8$



- **BVP:** $y(0) = 2, \quad y(2) = 4$

  $y(-2) = 5, \quad y'(3) = -2$