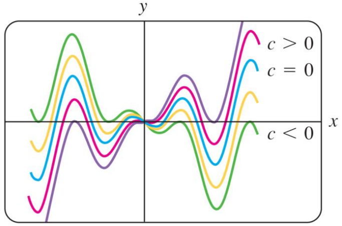
Fall 2019

微分方程
Differential Equations

Unit 01.1
Definitions and Terminology

$$\boxed{\frac{dy}{dx} = 0.2xy}$$

$$\boxed{y = \phi(x)}$$



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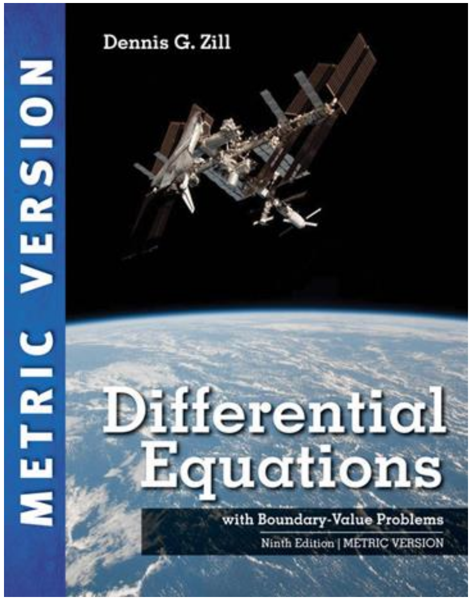


Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

- The derivative dy/dx of a function $y = \phi(x)$

  is itself another function $\phi'(x)$

  found by an appropriate rule.

- The exponential function $y = e^{0.1x^2}$

  is differentiable on the interval $(-\infty, \infty)$

  and by the Chain Rule

  its first derivative is $\frac{dy}{dx} = 0.2xe^{0.1x^2}$

or

$$\frac{dy}{dx} = (0.2\,x)\,y \quad ----- \quad (1)$$

Definition 1.1.1: Differential Equation (DE)

- An equation containing the derivatives

  of one or more unknown functions

  (or dependent variables),

  with respect to one or more independent variables,

  is said to be a differential equation (DE).

$$\frac{dy}{dx} = (0.2\,x)\,y$$

- We shall classify differential equations

  according to type, order, and linearity.

Classification by Type

- If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, it is said to be an ordinary differential equation (ODE).

$$\frac{dy}{dx} + 5y = e^x$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0 \quad \text{and}$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y \qquad --- \qquad (2)$$

- An ODE can contain more than one unknown function.

- An equation involving partial derivatives

  of one or more unknown functions

  of two or more independent variables

  is called a partial differential equation (PDE).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad --- \quad (3)$$

- *u* and *v* must be functions

  of *two or more* independent variables.

Notation

- Ordinary derivatives will be written by using

  either the **Leibniz notation** $dy/dx,\ d^2y/dx^2,\ d^3y/dx^3, \ldots$

  or the **prime notation** $y', y'', y''', \ldots.$
- The fourth derivative is written $y^{(4)}$ instead of y''''.

  In general, the nth derivative of y is written d^ny/dx^n or $y^{(n)}$.
- The Leibniz notation has an advantage

  over the prime notation in that it clearly displays

  both the dependent and independent variables.
- For example,

$$\frac{d^2x}{dt^2} + 16x = 0$$

- x now represents a dependent variable,

  whereas the independent variable is t.

Notation

- Newton's **dot notation** is sometimes used
  to denote derivatives with respect to time t.
- Differential equation $d^2s/dt^2 = -32$
  becomes $\ddot{s} = -32$.

- Partial derivatives are often denoted by **a subscript notation**, indicating the independent variables.
- For example, with the subscript notation,
  the second equation in (3) becomes $u_{xx} = u_{tt} - 2u_t$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} \quad --- \quad (3-2)$$

- $f \quad \longrightarrow \quad f(x)$ or $f(t)$
- $y \quad \longrightarrow \quad y(x)$ or $y(t)$

- **Leibniz notation**
  - $\dfrac{dy}{dx}, \quad \dfrac{d^2y}{dx^2}, \quad \dfrac{d^3y}{dx^3}, \quad \ldots \quad \dfrac{d^ny}{dx^n}$
- **Prime notation**
  - $y' \quad y'' \quad y''' \quad \ldots \quad y^{(n)}$
- **Dot notation**
  - $\dot{y} \quad \ddot{y} \quad \dddot{y} \quad \ddddot{y} \quad \ldots$
- **Subscript notation**
  - $y_x \quad y_{xx} \quad y_{xxx} \quad \ldots$

Classification by Order

- The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

- The equation is a second-order ordinary differential equation.

- A first-order ordinary differential equation is sometimes written in **the differential form**

$$M(x,y)dx + N(x,y)dy = 0.$$

- We can express an nth-order ordinary differential equation in one dependent variable by the general form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0, \qquad --- \ (4)$$

- where F is a real-valued function

  of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$.

- The differential equation

$$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \qquad --- \ (5)$$

- where f is a real-valued continuous function,

  is referred to as the **normal form** of (4).

- Thus when it suits our purposes,

  we shall use the normal forms

  to represent general first-order and second-order ODEs.

$$\frac{dy}{dx} = f(x, y)$$

$$and$$

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

- An nth-order ordinary differential equation (4)

  is said to be **linear** if *F* is linear in $y, y', \ldots, y^{(n)}$.

- This means that an nth-order ODE is linear when (4) is

  $$a_{n(x)} y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y - g(x) = 0$$

  or

  $$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \quad ---(6)$$

- Two important special cases of (6) are

  linear first-order (n = 1) and linear second order (n = 2) DEs:

  $$a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

  $$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \quad ---(7)$$

Classification by Linearity

- $a_{n(x)}y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$
- For the additive combination on the left-hand side,

  two characteristic properties of a linear ODE are as follows:
  - The dependent variable $\mathbf{y}$ and all its derivatives

    $\mathbf{y}', \mathbf{y}'', \ldots, \mathbf{y}^{(\mathbf{n})}$ are of the first degree,

    that is, the power of each term involving $\mathbf{y}$ is 1.
  - The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$

    depend at most on the independent variable $\mathbf{x}$.
- A nonlinear ODE is simply one that is not linear.
- Nonlinear functions of the dependent variable/its derivatives,

  such as $sin(y)$ or $e^{y'}$, cannot appear in a linear equation.

Definition 1.1.2: Solution of an ODE

- Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I,

  substituted into an *n*th-order ODE

  the equation is redced into an identity,

  the function is said to be a **solution** of the eqn on the interval.

- A solution of an *n*th-order ODE (4) is a function f that possesses at least n derivatives and for which

$$F\left(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)\right) = 0 \quad \textbf{for all } x \textbf{ in } I.$$

- The interval I in Definition 1.1.2 is variously called

  the **interval of definition**,

  the **interval of existence**,

  the **interval of validity**, or

  the **domain of the solution**

  and can be

  an open interval (a, b),

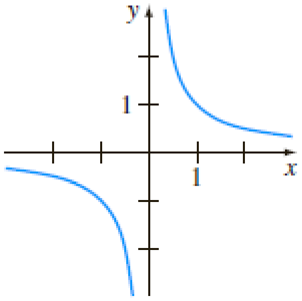
  a closed interval $[a, b]$,

  an infinite interval (a, ∞), and so on.

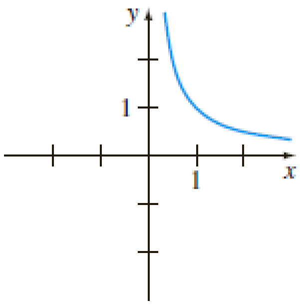
- A solution of a DE that is identically zero on an interval I,

  is said to be a **trivial solution**.

- The graph of a solution ϕ of an ODE is called a **solution curve.**
- Since ϕ is a differentiable function, it is continuous on its interval I of definition.
- Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ.
- The figures illustrate the difference.



(a) function $y = 1/x,\ x \neq 0$



(b) solution $y = 1/x,\ (0, \infty)$

$$y = 1/x$$

Figure 1.1.1: The function $y = 1/x$ is not the same as the solution $y = 1/x$

- A solution
  in which the dependent variable is expressed solely
  in terms of the independent variable and constants
  is said to be an **explicit solution.**
- Methods of solution do not always
  lead directly to an explicit solution $\boldsymbol{y = \phi(x)}$.
- This is particularly true
  when we attempt to solve nonlinear first-order DEs.
- Often we have to be content with
  a relation or expression $\boldsymbol{G(x, y) = 0}$
  that defines a solution $\boldsymbol{\phi}$ implicitly.

Definition 1.1.3: Implicit Solution of an ODE

- A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE (4) on an interval I, provided that there exists at least one function φ that satisfies the relation as well as the DE on I.

- The relation $x^2 + y^2 = 25$ is an implicit solution of the DE

$$\frac{dy}{dx} = -\frac{x}{y} \qquad ---(8)$$

on the open interval $(-5, 5)$.

Definition 1.1.3: Implicit Solution of an ODE

- The relation $x^2 + y^2 = 25$ is an implicit solution of the DE

$$\frac{dy}{dx} = -\frac{x}{y} \qquad ---(8)$$

on the open interval $(-5, 5)$.

- By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \quad \text{or} \quad 2x + 2y\frac{dy}{dx} = 0 \qquad ---(9)$$

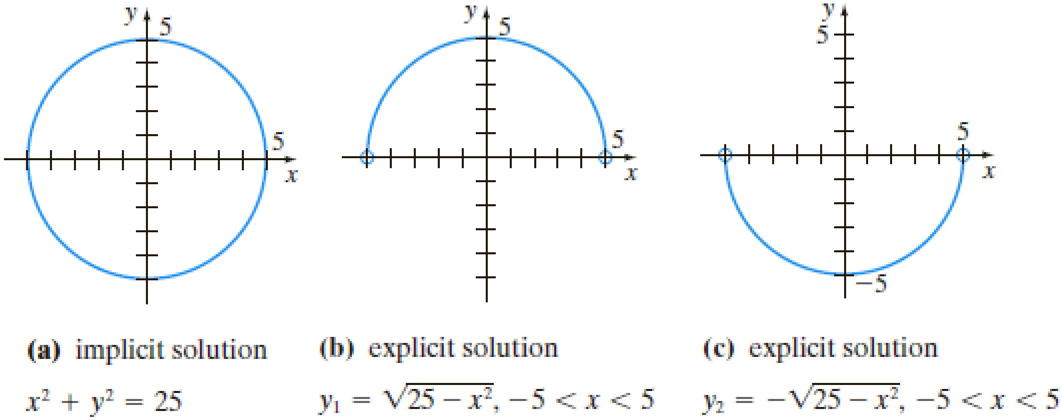


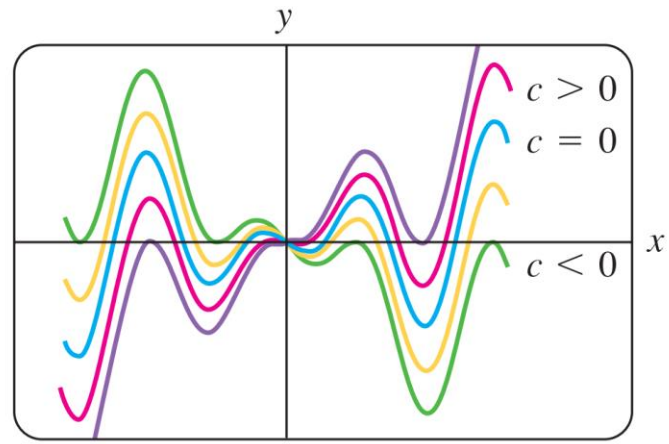
Figure 1.1.2: An implicit solution and two explicit solutions of (8)

- A solution of $F(x, y, y') = 0$ containing a constant c is a set of solutions $G(x, y, c) = 0$ called a **one-parameter family of solutions.**
- When solving an *n*th-order DE $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$ we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$.
- A single DE can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameter(s).
- A solution of a DE that is free of parameters is called a **particular solution.**

- The parameters in a family of solutions such as $\boldsymbol{G(x, y, c_1, c_2, \ldots, c_n) = 0}$ are *arbitrary* up to a point.

$$xy' - y = x^2 \sin x \text{ on the interval } (-\infty, \infty)$$

$$y = cx - x \cos x$$



- Sometimes a DE possesses a solution that is not a member of a family of solutions of the equation, that is, a solution that cannot be obtained by specializing *any* of the parameters in the family of solutions.
- Such an extra solution is called a **singular solution**.

- A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.
- For example, if x and y denote dependent variables and t denotes the independent variable, then a system of two first-order DEs is given by

$$\begin{aligned} \frac{dx}{dt} &= f(t,x,y) \\ \frac{dy}{dt} &= g(t,x,y) \end{aligned} \quad ---(10)$$

- A **solution** of a system such as (10)

  is a pair of differentiable functions

$$x = \phi_1(t),$$

$$y = \phi_2(t),$$

  defined on a common interval I,

  that satisfy each equation of the system on this interval.