Chapter 1

CONTROL SCIENCE AND FEEDBACK THEORY

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Abstract The discussion of **Control Science and Feedback Theory** is divided into three parts: (1) brief history of control; (2) classical control; (3) modern control. Brief histroy of control introduces the history of control science and feedback theory. Classical control covers the root-locus design methods and frequency-response design methods. Modern control inlcudes the disucssion of the state-space modelling and related control analysis and design methods such as state feedback and estimation, optimal control, adaptive control, and robust control etc.

Keywords: Root Locus, Frequency Response, Bode Plot, State-Space Model, Optimal Control, Adaptive Control, Robuts Control

1. Brief History of Control Science and Feedback Theory

2. Mathematical Models of Control Systems

3. Classical Control Methodolgies

3.1 The Root Locus Methods

3.2 Frequency Response Methods

4. Modern Control Methodologies

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*** Modern Control Analysis & Design by Feng-Li LIAN ***

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Most analysis and design approaches in modern control stem from the state-space mathematical model of physcial systems. Based on the state-space model, fundamental properties of a physical system such as stability, controllability, and observability, can be analyzed. These perperties are the key measures to characterize the system. If any of the system properties does not fulfill the requirement, further design methodologies are used to modify them. Generally speaking, design methodologies can be classified into two categories: fundamental and advanced. Fundamental design methodologies include state feedback, state estimation, and output feedback. Advanced desgin methodologies include optimal control: LQR and LQG, adaptive control, and robust control.

Key criteria of utilizing any of them are simply discussed as follows. Detailed discussions and related mathematical derivations can be in the following sections. If the system states are not stable, but controllable, they can be stabilized by a state feedback law. If the system states are not measureable, but observable, they can estimated by a state esimation agorithm. An output feedback approach tackles the problem of the above two cases. When the cost of the system states and inputs is subjected to certain weighting mechansim, optimal control is the major tool to handle this type of problems. Optimal control has two classes of methods: linear quadratic regulation (LQR) and linear quadratic guassion (LQG) that are for systems without and with random external inputs, respectively. When a system has some known uncertainty, both adaptive control and robust control can be applied. Adaptive control is mainly for the system with structured uncertainty, or unknown constant uncertainty; while robust control is mainly for the system with un-structured uncertainty, or unknown but bounded uncertainty.

In the following, we discuss fundamental system descriptions, properties of the general model, fundamental design methodologies, and advanced design methodologies.

4.1 Fundamental System Descriptions by State-Space Model

Modeling of Dynamcal Systems. In general, most dynamical systems in electrical engineering, mechanical engineering, chemical engineering, and other engineering disciplines, are modeled by a set of a finite number of coupled first-order ordinary differential equations:

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, x_2, \cdots, x_n, u_1, \cdots, u_p, v_1, \cdots, v_p) \\
\dot{x}_2 &= f_2(t, x_1, x_2, \cdots, x_n, u_1, \cdots, u_p, v_1, \cdots, v_p) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, x_2, \cdots, x_n, u_1, \cdots, u_p, v_1, \cdots, v_p),
\end{aligned}$$

and algebraic equations:

$$\begin{aligned}
y_1 &= h_1(t, x_1, x_2, \cdots, x_n, u_1, \cdots, u_p, w_1, \cdots, w_q) \\
y_2 &= h_2(t, x_1, x_2, \cdots, x_n, u_1, \cdots, u_p, w_1, \cdots, w_q) \\
\vdots \quad & \quad \vdots \\
y_q &= h_q(t, x_1, x_2, \cdots, x_n, u_1, \cdots, u_p, w_1, \cdots, w_q),
\end{aligned}$$

where $x_i, i = 1, \cdots n$, $y_j, j = 1, \cdots q$, $u_k, k = 1, \cdots p$, $w_j, j = 1, \cdots q$, $v_k, k = 1, \cdots p$, are the state, output, input, sensing noice variables, actuation disturbance of the system, and $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t.

For notation simplicity, define

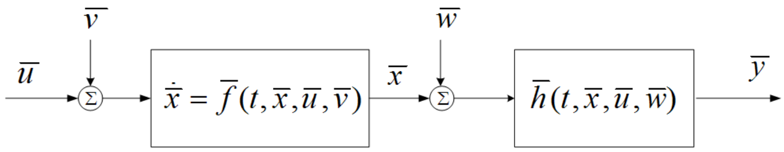
$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}, \; \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \; \bar{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{bmatrix}, \; \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_q \end{bmatrix},$$

$$\bar{f}(t, \bar{x}, \bar{u}, \bar{v}) = \begin{bmatrix} f_1(t, \bar{x}, \bar{u}, \bar{v}) \\ f_2(t, \bar{x}, \bar{u}, \bar{v}) \\ \vdots \\ f_n(t, \bar{x}, \bar{u}, \bar{v}) \end{bmatrix}, \qquad \bar{h}(t, \bar{x}, \bar{u}, \bar{w}) = \begin{bmatrix} h_1(t, \bar{x}, \bar{u}, \bar{w}) \\ h_2(t, \bar{x}, \bar{u}, \bar{w}) \\ \vdots \\ h_q(t, \bar{x}, \bar{u}, \bar{w}) \end{bmatrix},$$

and rewrite the set of differential and algebraic equations as follows:

$$\begin{aligned}
\dot{\bar{x}} &= \bar{f}(t, \bar{x}, \bar{u}, \bar{v}) \\
\bar{y} &= \bar{h}(t, \bar{x}, \bar{u}, \bar{w})
\end{aligned}$$

that are called the state and output equations, respectively, both together as the state-space model, or simply the state model. Normally, $\bar{w}$ and $\bar{v}$ are the vectors of the actuation disturbance and the sensing noice, respectively, and they are considered as any time-varying, or maybe time-invariant, uncertaint quantities affecting the true values of state, input, and output variables.



General system description

Figure 1.1. General system description.

Unforced Systems. If the input, disturbance, and noise to the system is zero, i.e., $\bar{u} = \bar{0}$, $\bar{u} = \bar{\gamma}(t)$, or $\bar{u} = \bar{\gamma}(\bar{x})$, and $\bar{w} = \bar{0}, \bar{v} = \bar{0}$, the type of systems are called *unforced systems* and can be described as follows:

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(t, \bar{x}) \\ \bar{y} &= \bar{h}(t, \bar{x}) \end{aligned}$$

Automonous or Time-Invariant Systems. Furthermore, if the function $\bar{f}$ does not depend explicitly on t, that is,

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) \\ \bar{y} &= \bar{h}(\bar{x}) \end{aligned}$$

the system is said to be autonomous or time-invariant.

Linear Time-Invariant Systems. If the system is linear, time-invariant, and $\bar{w} = \bar{0}, \bar{v} = \bar{0}$, i.e., without any disturbance and noise, the LTI model can be described as follows:

$$\begin{aligned} \dot{\bar{x}} &= \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u} \\ \bar{y} &= \bar{\bar{C}} \cdot \bar{x} + \bar{\bar{A}} \cdot \bar{u} \end{aligned}$$

where $\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}, \bar{\bar{D}}$ are constant matrices.

4.2 Properties of the general model

The key properties of a system are *stability, controllability, and observability.* Their definitions and the tools (theorems) to analyze these properties are discussed as follows.

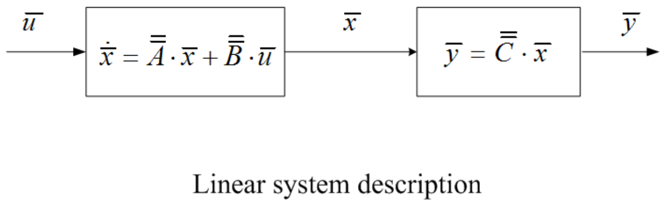


Figure 1.2. Linear system description.

Stability. The definition of stability is from the state point of view.

Definition 1: [Khalil 2002] [Chen 1999]

The equilibrium point $\bar{x} = \bar{0}$ of the autonomous system: $\dot{\bar{x}} = \bar{f}(\bar{x})$ is *stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$||\bar{x}(0)|| < \delta \Rightarrow ||\bar{x}(t)|| < \epsilon, \forall t \geq 0.$$

$\bar{x} = \bar{0}$ is *unstable* if it is *not stable.* $\bar{x} = \bar{0}$ is *asymptotically stable* if it is *stable* and δ can be chosen such that

$$||\bar{x}(\bar{0})|| < \delta \Rightarrow \lim_{t \to \infty} \bar{x}(t) = \bar{0}$$

That is, the autonomous system $\dot{\bar{x}} = \bar{f}(\bar{x})$ is *stable* if every finite intial state $\bar{x}_0$ excites a bounded response. It is *asymptotically stable* if every finite initial state excites a bounded response, which, in addition, approaches 0 as $t \to \infty$.

Theorems used to determine whether a system is stable or not.

Theorem 1: [Khalil 2002]

Let $\bar{x} = \bar{0}$ be an equlibrium point for the autonomous system and $D \subset R^n$ be a domain containing $\bar{x} = \bar{0}$. Let $V : D \to R$ be a continuously differentiable function such that

$$V(\bar{0}) = 0 \text{ and } V(\bar{x}) > 0 \text{ in } D - \{\bar{0}\}$$

$$\dot{V}(\bar{x}) \leq 0 \text{ in } D$$

Then, $\bar{x} = \bar{0}$ is stable. Moreover, if

$$\dot{V}(\bar{x}) < 0 \text{ in } D - \{\bar{0}\}$$

then $\bar{x} = \bar{0}$ is asymptotically stable.

For linear systems, the stability theorem can be stated as follows.

Theorem 2: [Chen 1999]

The equation $\dot{\bar{x}} = \bar{\bar{A}} \cdot \bar{x}$ is marginally stable if and only if all eigenvalues of $\bar{\bar{A}}$ have zero or negative real parts and those with zero real parts are simple roots of the minimal polynomial of $\bar{\bar{A}}$.

The equation $\dot{\bar{x}} = \bar{\bar{A}} \cdot \bar{x}$ is asymptotically stable if and only if all eigenvalues of $\bar{\bar{A}}$ have negative real parts.

Controllability. The definition of controllability.

Definition 2: [Chen 1999] [Saystry 1999]

The state equation is said to be *controllable* if for any initial state $\bar{x}_0$ and any final state $\bar{x}_f$, there exist a time t_f and an admissible input defined on $[t_0, t_f]$ that transfers $\bar{x}(t_0) = \bar{x}_0$ to $\bar{x}(t_f) = \bar{x}_f$. Otherwise, it is said to be *uncontrollable*.

The theorems to analyze the controllability condition of a nonlinear system is too complex. Here only the theorems for linear systems are stated.

Theorem 3: [Chen 1999]

The LTI system is *controllable* or ($\bar{\bar{A}}$, $\bar{\bar{B}}$ is a controllable pair if one of the following is satisfied.

1 The $n \times n$ matrix

$$\bar{\bar{W}}_c(t) = \int_{t_0}^{t} e^{\bar{\bar{A}}\tau} \cdot \bar{\bar{B}} \cdot \bar{\bar{B}}^{\mathsf{T}} \cdot e^{\bar{\bar{A}}^{\mathsf{T}}\tau} d\tau$$

is nonsingular for any $t > t_0$.

2 The $n \times np$ *controllability matrix*

$$\bar{\bar{\mathcal{C}}} = [\ \bar{\bar{B}} \ \ \bar{\bar{A}} \cdot \bar{\bar{B}} \ \ \bar{\bar{A}}^2 \cdot \bar{\bar{B}} \ \ \cdots \ \ \bar{\bar{A}}^{n-1} \cdot \bar{\bar{B}}\ ]$$

has rank n (full row rank).

3 The $n \times (n+p)$ matrix $[\bar{\bar{A}} - \lambda\bar{\bar{I}}, \ \ \bar{\bar{B}}\ ]$ has full row rank at every eigenvalue λ of $\bar{\bar{A}}$.

4 If, in addition, all eigenvalue of $\bar{\bar{A}}$ have negative real parts, then the unique solution of

$$\bar{\bar{A}} \cdot \bar{\bar{W}}_c + \bar{\bar{W}}_c \cdot \bar{\bar{A}}^{\mathsf{T}} = -\bar{\bar{B}} \cdot \bar{\bar{B}}^{\mathsf{T}}$$

is positive definite. The solution is called the *controllability Gramian* and can be expressed as

$$\bar{\bar{W}}_c = \int_0^{\infty} e^{\bar{\bar{A}}\tau} \cdot \bar{\bar{B}} \cdot \bar{\bar{B}}^{\mathsf{T}} \cdot e^{\bar{\bar{A}}^{\mathsf{T}}\tau} d\tau$$

What a controllable system can do, see state feedback section.

Observability. The definition of observability.

Definition 3: [Chen 1999]

The state equation is said to be *observable* if for any unknown initial state $\bar{x}_0$ there exist a finite time T such that the knowledge of the input $\bar{u}$ and the output $\bar{y}$ over $[0, T]$ suffices to determine uniquely the initial state $\bar{x}_0$. Otherwise, it is said to be *unobservable*.

The theorem to analyze the observability condition of a linear system is stated as follows.

Theorem 4: [Chen 1999]

The LTI system is *observable* or $(\bar{\bar{A}}, \bar{\bar{C}})$ is a observable pair if one of the following is satisfied.

1 The $n \times n$ matrix

$$\bar{\bar{W}}_o(t) = \int_0^{t} e^{\bar{\bar{A}}^{\mathsf{T}}\tau} \cdot \bar{\bar{C}}^{\mathsf{T}} \cdot \bar{\bar{C}} \cdot e^{\bar{\bar{A}}\tau} d\tau$$

is nonsingular for any $t > 0$.

2 The $nq \times n$ *observability matrix*

$$\bar{\bar{O}} = \begin{bmatrix} \bar{\bar{C}} \\ \bar{\bar{C}} \cdot \bar{\bar{A}} \\ \vdots \\ \bar{\bar{C}} \cdot \bar{\bar{A}}^{n-1} \end{bmatrix}$$

has rank n (full column rank).

3 The $(n + q) \times n$ matrix $\begin{bmatrix} \bar{\bar{A}} - \lambda \bar{\bar{I}} \\ \bar{\bar{C}} \end{bmatrix}$ has full column rank at every eigenvalue, λ, of $\bar{\bar{A}}$.

4 If, in addition, all eigenvalue of $\bar{\bar{A}}$ have negative real parts, then the unique solution of

$$\bar{\bar{A}}^{\mathsf{T}} \cdot \bar{\bar{W}}_o + \bar{\bar{W}}_o \cdot \bar{\bar{A}} = -\bar{\bar{C}}^{\mathsf{T}} \bar{\bar{C}}$$

is positive definite. The solution is called the *observability Gramian* and can be expressed as

$$\bar{\bar{W}}_o = \int_0^\infty e^{\bar{\bar{A}}^\mathsf{T}\tau} \cdot \bar{\bar{C}}^\mathsf{T} \cdot \bar{\bar{C}} \cdot e^{\bar{\bar{A}}\tau} d\tau$$

What a observable system can do, see state estimation section.

The controllability and obervability theorems show the duality between the pairs $(\bar{\bar{A}},\ \bar{\bar{B}})$ and $(\bar{\bar{A}},\ \bar{\bar{C}})$.

Kalman Canonical Decomposition. Based on the definition of controllability and observability, the overall system states $\bar{x}$ can decomposed into four parts, namely, both *controlable* and *observable* states $\bar{x}_{CO}$, *controlable* but *unobservable* states $\bar{x}_{C\bar{O}}$, *observable* but *uncontrolable* states $\bar{x}_{\bar{C}O}$, and neither *controlable* nor *observable* states $\bar{x}_{\bar{C}\bar{O}}$.

The followling is the Kalman decomposition theorem.

Theorem 5: [Chen 1999]

Every state-space model of LTI systems can be transformed, by an equivalence transformation, into the following canonical form:

$$\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_{CO} \\ \dot{\bar{x}}_{C\bar{O}} \\ \dot{\bar{x}}_{\bar{C}O} \\ \dot{\bar{x}}_{\bar{C}\bar{O}} \end{bmatrix} &= \begin{bmatrix} \bar{\bar{A}}_{CO} & \bar{\bar{0}} & \bar{\bar{A}}_{13} & \bar{\bar{A}} \\ \bar{\bar{A}}_{21} & \bar{\bar{A}}_{C\bar{O}} & \bar{\bar{A}}_{23} & \bar{\bar{A}}_{24} \\ \bar{\bar{0}} & \bar{\bar{0}} & \bar{\bar{A}}_{\bar{C}O} & \bar{\bar{0}} \\ \bar{\bar{A}} & \bar{\bar{0}} & \bar{\bar{A}}_{43} & \bar{\bar{A}}_{\bar{C}\bar{O}} \end{bmatrix} \begin{bmatrix} \bar{x}_{CO} \\ \bar{x}_{C\bar{O}} \\ \bar{x}_{\bar{C}O} \\ \bar{x}_{\bar{C}\bar{O}} \end{bmatrix} + \begin{bmatrix} \bar{\bar{B}}_{CO} \\ \bar{\bar{B}}_{C\bar{O}} \\ \bar{\bar{0}} \\ \bar{\bar{0}} \end{bmatrix} \bar{u} \\
\bar{y} &= \begin{bmatrix} \bar{\bar{C}}_{CO} & \bar{\bar{0}} & \bar{\bar{C}}_{\bar{C}O} & \bar{\bar{0}} \end{bmatrix} \begin{bmatrix} \bar{x}_{CO} \\ \bar{x}_{C\bar{O}} \\ \bar{x}_{\bar{C}O} \\ \bar{x}_{\bar{C}\bar{O}} \end{bmatrix} + \bar{\bar{D}}\bar{u}
\end{aligned}$$

Furthermore, the state equation is zero-state equivalent to the controllable and observable state equation

$$\begin{aligned}
\dot{\bar{x}}_{CO} &= \bar{\bar{A}}_{CO} \cdot \bar{x}_{CO} + \bar{\bar{B}}_{CO} \cdot \bar{u} \\
\bar{y} &= \bar{\bar{C}}_{CO} \cdot \bar{x}_{CO} + \bar{\bar{D}} \cdot \bar{u}
\end{aligned}$$

and has the transfer function matrix

$$\hat{\bar{\bar{G}}}(s) = \bar{\bar{C}}_{CO} \cdot (s\bar{\bar{I}} - \bar{\bar{A}}_{CO})^{-1} \cdot \bar{\bar{B}}_{CO} + \bar{\bar{D}}$$

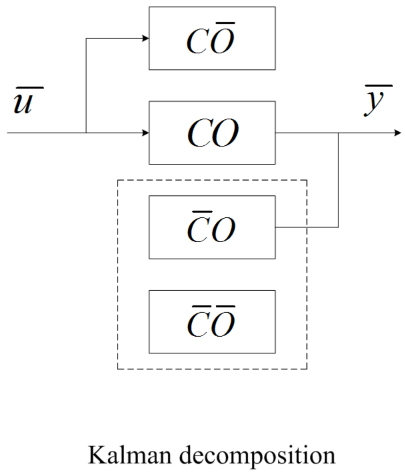


Figure 1.3. Kalman decomposition.

Stabilizability. The definition of stabilizability.

Definition 2: [Chen 1999]
The state equation is said to be *stabilizable* if the state equation is not controllable and the unstable subspace is controllable. That is, the uncontrollable subspace should be stable. Hence, a controlable state equation is stabilizable.

4.3 Fundamental Design Methodologies

State Feedback. Consider a system described by the following state-space equation

$$\begin{aligned} \dot{\bar{x}} &= \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u} \\ \bar{y} &= \bar{\bar{C}} \cdot \bar{x} \end{aligned}$$

In state feedback, the input **u** is given by

$$\bar{u} = \bar{r} - \bar{\bar{K}} \cdot \bar{x}$$

where $\bar{\bar{K}}$ is a $p \times n$ real constant matrix and $\bar{r}$ is a reference signal. Then, substituting $\bar{u}$ into the state-space equation, yields

$$\dot{\bar{x}} = (\bar{\bar{A}} - \bar{\bar{B}} \cdot \bar{\bar{K}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{r} \quad (1.1)$$
$$\bar{y} = \bar{\bar{C}} \cdot \bar{x} \quad (1.2)$$

Then, all eigenvalues of $(\bar{\bar{A}} - \bar{\bar{B}} \cdot \bar{\bar{K}})$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant $\bar{\bar{K}}$ if and only if $(\bar{\bar{A}},\ \bar{\bar{B}})$ is *controllable*.

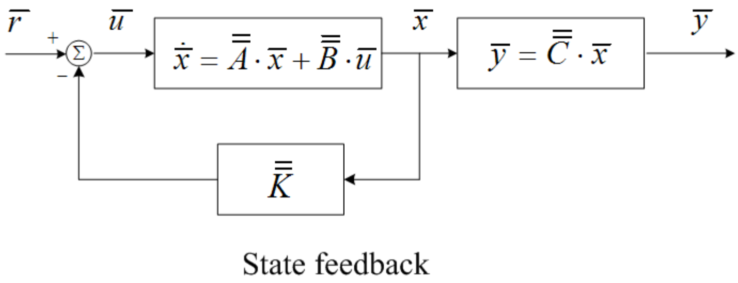


Figure 1.4. State feedback.

(Deterministic) State Estimation. The problem is to use available input singal $\bar{u}$ and output signal $\bar{y}$ as the new input to a state-estimation system whose output gives an estimate of the state $\bar{x}$. Consider a system described by the following state-space equation

$$\dot{\bar{x}} = \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u}$$
$$\bar{y} = \bar{\bar{C}} \cdot \bar{x}$$

Design an full-dimensional state estimator described as follows:

$$\begin{aligned} \dot{\bar{x}}_e &= \bar{\bar{A}} \cdot \bar{x}_e + \bar{\bar{B}} \cdot \bar{u} + \bar{\bar{L}} \cdot (\bar{y} - \bar{\bar{C}} \cdot \bar{x}_e) \\ &= (\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}}) \cdot \bar{x}_e + \bar{\bar{B}} \cdot \bar{u} + \bar{\bar{L}} \cdot \bar{y} \end{aligned}$$

where $\bar{x}_e$ is an estimate of the true state $\bar{x}$ and $\bar{\bar{L}}$ is a $n \times q$ real constant matrix.

Define the error vector as

$$\bar{e} = \bar{x} - \bar{x}_e$$

The error dynamics can experessed as follows:

$$\dot{\bar{e}} = (\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}}) \cdot \bar{e}$$

Then, all eigenvalues of $(\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}})$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant $\bar{\bar{L}}$ if and only if $(\bar{\bar{A}}, \bar{\bar{C}})$ is *controllable.*

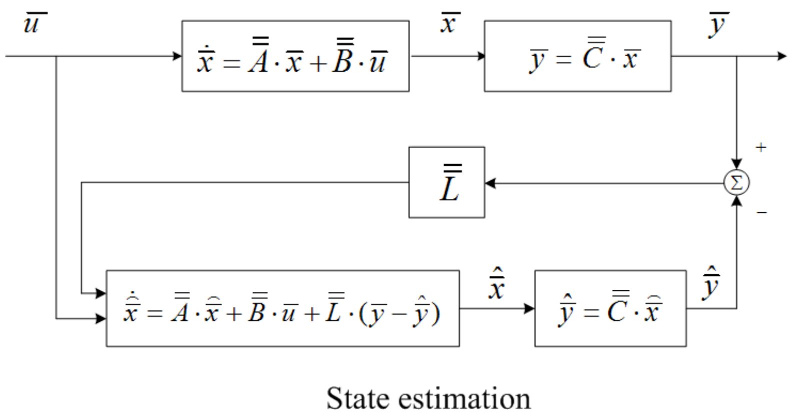


Figure 1.5. State estimation.

Output Feedback. When the system state $\bar{x}$ is not available for state feedback, the problem is to use available input singal $\bar{u}$ and output signal $\bar{y}$ to estimate $\bar{x}$. Then, use the state estimate $\bar{x}_e$ to design state feedback. This is called *output feedback.*

Consider a system described by the following state-space equation

$$\begin{aligned} \dot{\bar{x}} &= \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u} \\ \bar{y} &= \bar{\bar{C}} \cdot \bar{x} \end{aligned}$$

Design an full-dimensional state estimator described as follows:

$$\dot{\bar{x}}_e = (\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}}) \cdot \bar{x}_e + \bar{\bar{B}} \cdot \bar{u} + \bar{\bar{L}} \cdot \bar{y}$$

and implement the state feedback law as follows:

$$\bar{u} = \bar{r} - \bar{\bar{A}} \cdot \bar{x}_e$$

The overall system can be described as follows:

$$
\begin{aligned}
\dot{\bar{x}} &= \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u} \\
\bar{y} &= \bar{\bar{C}} \cdot \bar{x} \\
\dot{\bar{x}}_e &= (\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}}) \cdot \bar{x}_e + \bar{\bar{B}} \cdot (\bar{r} - \bar{\bar{K}} \cdot \bar{x}_e) + \bar{\bar{L}} \cdot \bar{\bar{C}} \cdot \bar{x}
\end{aligned}
$$

They can be combined as

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{x}}_e \end{bmatrix} &= \begin{bmatrix} \bar{\bar{A}} & -\bar{\bar{B}} \cdot \bar{\bar{K}} \\ \bar{\bar{L}} \cdot \bar{\bar{C}} & \bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}} - \bar{\bar{B}} \cdot \bar{\bar{K}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{\bar{x}} \end{bmatrix} + \begin{bmatrix} \bar{\bar{B}} \\ \bar{\bar{B}} \end{bmatrix} \bar{r} \\
\bar{y} &= \begin{bmatrix} \bar{\bar{C}} & \bar{\bar{0}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{x}_e \end{bmatrix}
\end{aligned}
$$

Introduce the following equivalence transformation:

$$
\begin{bmatrix} \bar{x} \\ \bar{e} \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{x} - \bar{x}_e \end{bmatrix} = \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{0}} \\ \bar{\bar{I}} & -\bar{\bar{I}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{x}_e \end{bmatrix}
$$

and obtain the following equivalent state equation:

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{e}} \end{bmatrix} &= \begin{bmatrix} \bar{\bar{A}} - \bar{\bar{B}} \cdot \bar{\bar{K}} & \bar{\bar{B}} \cdot \bar{\bar{K}} \\ \bar{\bar{0}} & \bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{e} \end{bmatrix} + \begin{bmatrix} \bar{\bar{B}} \\ \bar{\bar{0}} \end{bmatrix} \bar{r} \\
\bar{y} &= \begin{bmatrix} \bar{\bar{C}} & \bar{\bar{0}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{e} \end{bmatrix}
\end{aligned}
$$

Hence, the eigenvalues of the system matrix of the overall system are the union of those of $\bar{\bar{A}} - \bar{\bar{B}} \cdot \bar{\bar{K}}$ and $\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}}$. Thus, inserting the state estimator does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the state estimator affected by the connection. Therefore, teh design of state feedback and the design of the state estimator can be carried out independently. This is the so-called the *separation principle* of the estimator-controller design procedure.

4.4 Advanced Design Methodologies

Optimal Control: Linear Quadratic Regulation (LQR) Method (Optimal State Feedback Gain). Two of key necessities of designing a controller for a system are to bound the magnitude of the various state variables in the system by practical consideration and to keep some measure of control magnitude bounded or even small during the course of a control action [4].

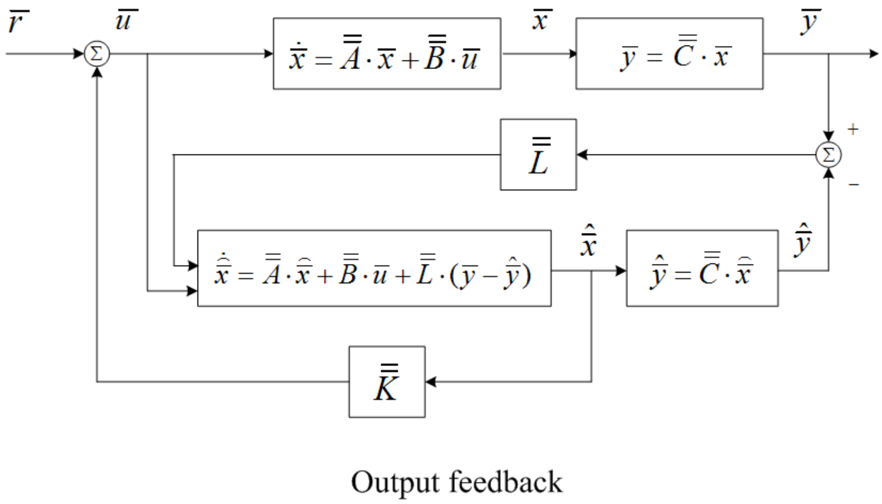


Figure 1.6. Output feedback.

One way to achieve the design goal is to set up a quadratic performance index $J(\bar{x}(t_0), \bar{u}(\cdot), t_0)$, and find a control law $\bar{u}^*(t)$ which minimizes it.

Consider a system described by the following state-space equation

$$\begin{aligned} \dot{\bar{x}} &= \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u} \\ \bar{y} &= \bar{\bar{C}} \cdot \bar{x} \end{aligned}$$

Then, the quadratic performance index is defined as:

$$\begin{aligned} J(\bar{x}(t_0), \bar{u}(\cdot), t_0) &= \int_{t_0}^{t_f} \left(\bar{u}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}(t) \cdot \bar{u}(t) + \bar{x}^{\mathsf{T}}(t) \cdot \bar{\bar{Q}}(t) \cdot \bar{x}(t) \right) dt \\ &+ \bar{x}^{\mathsf{T}}(t_f) \cdot \bar{\bar{Q}}_f \cdot \bar{x}(t_f) \end{aligned}$$

where $\bar{\bar{R}}(t) = \bar{\bar{R}}^{\mathsf{T}}(t) > 0, \bar{\bar{Q}}(t) = \bar{\bar{Q}}^{\mathsf{T}}(t) \geq 0, \forall t \geq t_0$, are continuously differential weighting matrices and $\bar{\bar{Q}}_f$ is a constant matrix.

The optimal control law minimizing the designated quadratic performance index is as:

$$\bar{u}^*(t) = -\bar{\bar{R}}^{-1}(t) \cdot \bar{\bar{B}}^{\mathsf{T}}(t) \cdot \bar{\bar{P}}(t) \cdot \bar{x}(t)$$

where $\bar{\bar{P}}(t)$ is symmetric and the solution of the following *matrix Riccati equation*:

$$
\begin{aligned}
-\dot{\bar{\bar{P}}}(t) &= \bar{\bar{P}}(t)\cdot\bar{\bar{A}}(t)+\bar{\bar{A}}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)-\bar{\bar{P}}(t)\cdot\bar{\bar{B}}(t)\cdot\bar{\bar{R}}^{-1}(t)\cdot\bar{\bar{B}}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)+\bar{\bar{Q}}(t)\\
\text{with} \quad & \bar{\bar{P}}(t_f)=\bar{\bar{Q}}(t_f)
\end{aligned}
$$

And the optimum performance index can be described as

$$
J^*(\bar{x}(t),t) = \bar{x}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)\cdot\bar{x}(t)
$$

Derivation of LQR Method. The optimal solution for the quadratic performance index is assumed to have the form [4]:

$$
J^*(\bar{x}(t),t) = \bar{x}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)\cdot\bar{x}(t)
$$

for some matrix $\bar{\bar{P}}(t)$, with loss of generality symmetric. If $\bar{\bar{P}}(t)$ is not symmetric, it may be replaced by the symmetric matrix $\frac{1}{2}\cdot[\bar{\bar{P}}(t)+\bar{\bar{P}}^{\mathsf{T}}(t)]$.

From the Hamilton-Jacobi equation:

$$
\frac{\partial J^*(\bar{x}(t),t)}{\partial t} = -\lim_{\bar{u}(t)}\left\{ l(\bar{x}(t),\bar{u}(t),t)+\left[\frac{\partial J^*(\bar{x}(t),t)}{\partial \bar{x}}\right]^{\mathsf{T}}\cdot\bar{f}(\bar{x}(t),\bar{u}(t),t)\right\}
$$

where

$$
\begin{aligned}
l(\bar{x}(t),\bar{u}(t),t) &= \bar{u}^{\mathsf{T}}(t)\cdot\bar{\bar{R}}(t)\cdot\bar{u}(t)+\bar{x}^{\mathsf{T}}(t)\cdot\bar{\bar{Q}}(t)\cdot\bar{x}(t),\\
\left[\frac{\partial J^*(\bar{x}(t),t)}{\partial \bar{x}}\right]^{\mathsf{T}} &= 2\cdot\bar{x}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t),\\
\text{and } \bar{f}(\bar{x}(t),\bar{u}(t),t) &= \bar{\bar{A}}(t)\cdot\bar{x}(t)+\bar{\bar{B}}(t)\cdot\bar{u}(t)
\end{aligned}
$$

Therefore,

$$
\bar{x}^{\mathsf{T}}\cdot\dot{\bar{\bar{P}}}\cdot\bar{x} = -\lim_{\bar{u}(t)}\left\{\bar{u}^{\mathsf{T}}\cdot\bar{\bar{R}}\cdot\bar{u}+\bar{x}^{\mathsf{T}}\cdot\bar{\bar{Q}}\cdot\bar{x}+2\cdot\bar{x}^{\mathsf{T}}\cdot\bar{\bar{P}}\cdot\bar{\bar{A}}\cdot\bar{x}+2\cdot\bar{x}^{\mathsf{T}}\cdot\bar{\bar{P}}\cdot\bar{\bar{B}}\cdot\bar{u}\right\}
$$

The expression on the right-hand side is:

$$
\begin{aligned}
\{\ \cdots\ \} &= (\bar{u}+\bar{\bar{R}}^{-1}\cdot\bar{\bar{B}}^{\mathsf{T}}\cdot\bar{\bar{P}}\cdot\bar{x})^{\mathsf{T}}\cdot\bar{\bar{R}}\cdot(\bar{u}+\bar{\bar{R}}^{-1}\cdot\bar{\bar{B}}^{\mathsf{T}}\cdot\bar{\bar{P}}\cdot\bar{x})\\
&+\ \bar{x}^{\mathsf{T}}\cdot(\bar{\bar{Q}}-\bar{\bar{P}}\cdot\bar{\bar{B}}\cdot\bar{\bar{R}}^{-1}\cdot\bar{\bar{B}}^{\mathsf{T}}\cdot\bar{\bar{P}}+\bar{\bar{P}}\cdot\bar{\bar{A}}+\bar{\bar{A}}^{\mathsf{T}}\cdot\bar{\bar{P}})\cdot\bar{x}
\end{aligned}
$$

Because $\bar{\bar{R}}$ is positive definite, the minimum solution is:

$$
\bar{u}^*(t) = -\bar{\bar{R}}^{-1}(t)\cdot\bar{\bar{B}}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)\cdot\bar{x}(t)
$$

Therefore, one can obtain:

$$
\bar{x}\cdot\dot{\bar{\bar{P}}}\cdot\bar{x} = -\bar{x}\left[\bar{\bar{Q}}-\bar{\bar{P}}\cdot\bar{\bar{B}}\cdot\bar{\bar{R}}^{-1}\cdot\bar{\bar{B}}^{\mathsf{T}}\cdot\bar{\bar{P}}+\bar{\bar{P}}\cdot\bar{\bar{A}}+\bar{\bar{A}}^{\mathsf{T}}\cdot\bar{\bar{P}}\right]\cdot\bar{x}
$$

This equation holds for all $\bar{x}$; therefore,

$$-\dot{\bar{\bar{P}}} = \bar{\bar{P}} \cdot \bar{\bar{A}} + \bar{\bar{A}}^{\mathsf{T}} \cdot \bar{\bar{P}} - \bar{\bar{P}} \cdot \bar{\bar{B}} \cdot \bar{\bar{R}}^{-1} \cdot \bar{\bar{B}}^{\mathsf{T}} \cdot \bar{\bar{P}} + \bar{\bar{Q}}$$

which is the matrix Riccati equation with the boundary condition

$$\bar{\bar{P}}(t_f) = \bar{\bar{Q}}_f.$$

Optimal Control: Linear Quadratic Guassion (LQG) Method (Optimal Statistical State Estimation). If both the actuation disturbance and sensing noise are present, the system can be described as [4]:

$$\begin{aligned}\dot{\bar{x}} &= \bar{\bar{A}} \cdot \bar{x} + \bar{\bar{B}} \cdot \bar{u} + \bar{v} \\ \bar{y} &= \bar{\bar{C}} \cdot \bar{x} + \bar{w}\end{aligned}$$

Suppose that $\bar{v}(\cdot)$, $\bar{w}(\cdot)$, $\bar{x}(t_0)$ are indepedently and gaussian with

$$\begin{aligned}
E\left\{\bar{v}(t) \cdot \bar{v}^{\mathsf{T}}(\tau)\right\} &= \bar{\bar{Q}}_e(t)\delta(t-\tau), \qquad E\left\{\bar{v}(t)\right\} \equiv \bar{0}, \\
E\left\{\bar{w}(t) \cdot \bar{w}^{\mathsf{T}}(\tau)\right\} &= \bar{\bar{R}}_e(t)\delta(t-\tau), \qquad E\left\{\bar{w}(t)\right\} \equiv \bar{0}, \\
E\left\{\left(\bar{x}(t_0) - \bar{m}\right) \cdot \left(\bar{x}(t_0) - \bar{m}\right)^{\mathsf{T}}\right\} &= \bar{\bar{P}}_{e0}, \qquad E\left\{\bar{x}(t_0)\right\} \equiv \bar{m}, \\
E\left\{\bar{x}(t_0) \cdot \bar{v}^{\mathsf{T}}(t)\right\} = E\left\{\bar{x}(t_0) \cdot \bar{w}^{\mathsf{T}}(t)\right\} &= \bar{\bar{0}}, \qquad \forall t, \\
E\left\{\bar{v}(t) \cdot \bar{w}^{\mathsf{T}}(t)\right\} &= \bar{\bar{0}}, \qquad \forall t, \tau
\end{aligned}$$

where $\bar{\bar{Q}}_e(t) \geq 0$, $\bar{\bar{R}}_e(t) > 0$.

Since the exact value of the state variables $\bar{x}$ is not measurable, the state should be estimated by the following dynamic equation:

$$\dot{\bar{x}}_e = (\bar{\bar{A}} - \bar{\bar{L}} \cdot \bar{\bar{C}}) \cdot \bar{x}_e + \bar{\bar{B}} \cdot \bar{u} + \bar{\bar{L}} \cdot \bar{y}, \qquad \bar{x}_e(t_0) = \bar{m}$$

where the estimation gain $\bar{\bar{L}}$ is given from

$$\bar{\bar{L}} = \bar{\bar{P}}_e(t) \cdot \bar{\bar{C}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}_e^{-1}(t)$$

where $\bar{\bar{P}}_e(t)$ is symmetric nonnegative definite and the solution of the following *matrix Riccati equation*:

$$\begin{aligned}\dot{\bar{\bar{P}}}_e(t) &= \bar{\bar{P}}_e(t) \cdot \bar{\bar{A}}^{\mathsf{T}}(t) + \bar{\bar{A}}(t) \cdot \bar{\bar{P}}_e(t) - \bar{\bar{P}}_e(t) \cdot \bar{\bar{C}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}_e^{-1}(t) \cdot \bar{\bar{C}}(t) \cdot \bar{\bar{P}}_e(t) + \bar{\bar{Q}}_e(t) \\ \text{with} \quad & \bar{\bar{P}}_e(t_0) = \bar{\bar{P}}_{e0}\end{aligned}$$

and minimizes the error covariance

$$E\left\{\left(\bar{x}(t)-\bar{x}_e(t)\right)\cdot\left(\bar{x}(t)-\bar{x}_e(t)\right)^{\mathsf{T}}\right\}$$

For the controller design, the following performance index is considered:

$$J(\bar{x}(t_0),\bar{u}(\cdot),t_0) \;=\; E\left\{\int_{t_0}^{t_f}\left(\bar{u}^{\mathsf{T}}(t)\cdot\bar{\bar{R}}(t)\cdot\bar{u}(t)+\bar{x}^{\mathsf{T}}(t)\cdot\bar{\bar{Q}}(t)\cdot\bar{x}(t)\right)dt\right\}$$

where the expectation is over $\bar{x}(t_0)$ and the processes $\bar{v}(\cdot)$ and $\bar{w}(\cdot)$ on the interval $[t_0,t_f]$.

The state feedback optimal control is then designed based on the LQR approach as

$$\bar{u}^*(t) \;=\; -\bar{\bar{R}}^{-1}(t)\cdot\bar{\bar{B}}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)\cdot\bar{x}_e(t)$$

where $\bar{\bar{P}}(t)$ is symmetric and the solution of the following *matrix Riccati equation*:

$$\begin{aligned}-\dot{\bar{\bar{P}}}(t) &= \bar{\bar{P}}(t)\cdot\bar{\bar{A}}(t)+\bar{\bar{A}}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)-\bar{\bar{P}}(t)\cdot\bar{\bar{B}}(t)\cdot\bar{\bar{R}}^{-1}(t)\cdot\bar{\bar{B}}^{\mathsf{T}}(t)\cdot\bar{\bar{P}}(t)+\bar{\bar{Q}}(t)\\ \text{with}\quad & \bar{\bar{P}}(T)=\bar{\bar{Q}}(T)\end{aligned}$$

Note that the estimate $\hat{\bar{x}}(t)$ of the true state $\bar{x}(t)$ is used. Again, the state feedback and estimation can be designed independently due to the separation principle.

Derivation of LQG Method. The estimation is a minimum variance estimate, that is, to construct from a measurement of $\bar{y}(t), t_0 \le t \le t_1$, such that

$$E\left\{\left(\bar{x}(t_1)-\bar{x}_e(t_1)\right)^{\mathsf{T}}\cdot\left(\bar{x}(t_1)-\bar{x}_e(t_1)\right)\right\}$$

is minimum [4]. Since all the random processes and variables are gaussian, and have zero mean, the vector $\bar{x}_e(t)$ can be derived by linear operations on $\bar{y}(t), t_0 \le t \le t_1$, that is, there exists some matrix function $\bar{\bar{M}}(t;t_1), t_0 \le t \le t_1$, such that

$$\bar{x}_e(t_1) \;=\; \int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t;t_1)\cdot\bar{y}(t)\,dt$$

Introduce a new square matrix function of time $\bar{\bar{Z}}(\cdot)$, of the same row dimension as $\bar{x}(\cdot)$. This function is defined from $\bar{\bar{M}}(\cdot)$ via the equation

$$\frac{d}{dt}\bar{\bar{Z}}(t) \;=\; -\bar{\bar{A}}^{\mathsf{T}}(t)\cdot\bar{\bar{Z}}(t)+\bar{\bar{C}}^{\mathsf{T}}(t)\cdot\bar{\bar{M}}(t),\qquad \bar{\bar{Z}}(t_1)=\bar{\bar{I}}$$

Furthermore, in the follow,

$$\begin{aligned}
\frac{d}{dt}\left[\bar{\bar{Z}}^{\mathsf{T}}(t)\cdot\bar{x}(t)\right] &= \dot{\bar{\bar{Z}}}^{\mathsf{T}}(t)\cdot\bar{x}(t)+\bar{\bar{Z}}^{\mathsf{T}}(t)\cdot\dot{\bar{x}}(t) \\
&= -\bar{\bar{Z}}^{\mathsf{T}}\cdot\bar{\bar{A}}\cdot\bar{x}+\bar{\bar{M}}^{\mathsf{T}}\cdot\bar{\bar{C}}\cdot\bar{x}+\bar{\bar{Z}}^{\mathsf{T}}\cdot\bar{\bar{A}}\cdot\bar{x}+\bar{\bar{Z}}^{\mathsf{T}}\cdot\bar{v} \\
&= \bar{\bar{M}}^{\mathsf{T}}\cdot\bar{y}-\bar{\bar{M}}^{\mathsf{T}}\cdot\bar{w}+\bar{\bar{Z}}^{\mathsf{T}}\cdot\bar{v}
\end{aligned}$$

Integrating this equation from t_0 to t_1, using the boundary condition on $\bar{\bar{Z}}$, leads to

$$\begin{aligned}
\bar{x}(t_1)-\bar{\bar{Z}}^{\mathsf{T}}(t_0)\cdot\bar{x}(t_0) &= \int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{y}(t)\,dt-\int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{w}(t)\,dt \\
&\quad +\int_{t_0}^{t_1}\bar{\bar{Z}}^{\mathsf{T}}(t)\cdot\bar{v}(t)\,dt
\end{aligned}$$

or

$$\begin{aligned}
\bar{x}(t_1)-\int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{y}(t)\,dt &= \bar{\bar{Z}}^{\mathsf{T}}(t_0)\cdot\bar{x}(t_0)-\int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{w}(t)\,dt \\
&\quad +\int_{t_0}^{t_1}\bar{\bar{Z}}^{\mathsf{T}}(t)\cdot\bar{v}(t)\,dt
\end{aligned}$$

Therefore, the following results

$$\begin{aligned}
& E\left\{\left[\bar{x}(t_1)-\int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{y}(t)\,dt\right]\cdot\left[\bar{x}(t_1)-\int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{y}(t)\,dt\right]^{\mathsf{T}}\right\} \\
= \; & E\left\{\bar{\bar{Z}}^{\mathsf{T}}(t_0)\cdot\bar{x}(t_0)\cdot\bar{x}^{\mathsf{T}}(t_0)\cdot\bar{\bar{Z}}(t_0)\right\} \\
& + \; E\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1}\bar{\bar{M}}^{\mathsf{T}}(t)\cdot\bar{w}(t)\cdot\bar{w}^{\mathsf{T}}(\tau)\cdot\bar{\bar{M}}(\tau)\,dt\,d\tau\right\} \\
& + \; E\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1}\bar{\bar{Z}}^{\mathsf{T}}(t)\cdot\bar{v}(t)\cdot\bar{v}^{\mathsf{T}}(\tau)\cdot\bar{\bar{Z}}(\tau)\,dt\,d\tau\right\}
\end{aligned}$$

because of the independence of $\bar{x}(t_0),\bar{w}(t),\bar{v}(t)$.

$\bar{\bar{Z}}(\cdot)$ and $\bar{\bar{M}}(\cdot)$ are unknown but deterministic. Hence, the three terms of the above equation are derived as follows, respectively.

$$\begin{aligned}
(1) &= \bar{\bar{Z}}^{\mathsf{T}}(t_0) \cdot E\left\{\bar{x}(t_0) \cdot \bar{x}^{\mathsf{T}}(t_0)\right\} \cdot \bar{\bar{Z}}(t_0) \\
&= \bar{\bar{Z}}^{\mathsf{T}}(t_0) \cdot \bar{\bar{P}}_{e0} \cdot \bar{\bar{Z}}(t_0) \\
(2) &= \int_{t_0}^{t_1} \int_{t_0}^{t_1} \bar{\bar{M}}^{\mathsf{T}}(t) \cdot E\left\{\bar{w}(t) \cdot \bar{w}^{\mathsf{T}}(\tau)\right\} \cdot \bar{\bar{M}}(\tau)\, dt\, d\tau \\
&= \int_{t_0}^{t_1} \int_{t_0}^{t_1} \bar{\bar{M}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}(t) \cdot \delta(t-\tau) \cdot \bar{\bar{M}}(\tau)\, dt\, d\tau \\
&= \int_{t_0}^{t_1} \bar{\bar{M}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}(t) \cdot \bar{\bar{M}}(t)\, dt \\
(3) &= \int_{t_0}^{t_1} \int_{t_0}^{t_1} \bar{\bar{Z}}^{\mathsf{T}}(t) \cdot E\left\{\bar{v}(t) \cdot \bar{v}^{\mathsf{T}}(\tau)\right\} \cdot \bar{\bar{Z}}(\tau)\, dt\, d\tau \\
&= \int_{t_0}^{t_1} \int_{t_0}^{t_1} \bar{\bar{Z}}^{\mathsf{T}}(t) \cdot \bar{\bar{Q}}(t) \cdot \delta(t-\tau) \cdot \bar{\bar{Z}}(\tau)\, dt\, d\tau \\
&= \int_{t_0}^{t_1} \bar{\bar{Z}}^{\mathsf{T}}(t) \cdot \bar{\bar{Q}}(t) \cdot \bar{\bar{Z}}(t)\, dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\left\{\Big[\bar{x}(t_1) - \bar{x}_e(t_1)\Big] \cdot \Big[\bar{x}(t_1) - \bar{x}_e(t_1)\Big]^{\mathsf{T}}\right\} \\
= \;& \bar{\bar{Z}}^{\mathsf{T}}(t_0) \cdot \bar{\bar{P}}_{e0} \cdot \bar{\bar{Z}}(t_0) + \int_{t_0}^{t_1} \Big[\bar{\bar{M}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}(t) \cdot \bar{\bar{M}}(t) + \bar{\bar{Z}}^{\mathsf{T}}(t) \cdot \bar{\bar{Q}}(t) \cdot \bar{\bar{Z}}(t)\Big]\, dt
\end{aligned}$$

Therefore, compared with the LQR method, the optimal solution for $\bar{\bar{M}}$ is

$$\bar{\bar{M}}^* = \bar{\bar{R}}^{-1}(t) \cdot \bar{\bar{C}}(t) \cdot \bar{\bar{P}}_e(t) \cdot \bar{\bar{Z}}(t)$$

where $\bar{\bar{P}}_e(t)$ is the solution of the Riccati equation

$$\begin{aligned}
\dot{\bar{\bar{P}}}_e(t) &= \bar{\bar{P}}_e(t) \cdot \bar{\bar{A}}^{\mathsf{T}}(t) + \bar{\bar{A}}(t) \cdot \bar{\bar{P}}_e(t) - \bar{\bar{P}}_e(t) \cdot \bar{\bar{C}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}_e^{-1}(t) \cdot \bar{\bar{C}}(t) \cdot \bar{\bar{P}}_e(t) + \bar{\bar{Q}}_e(t) \\
\text{with} \quad & \bar{\bar{P}}_e(t_0) = \bar{\bar{P}}_{e0}
\end{aligned}$$

The estimate is as follows

$$\begin{aligned}
\bar{x}_e(t_1) &= \int_{t_0}^{t_1} \bar{\bar{M}}^{\mathsf{T}}(t; t_0) \cdot \bar{y}(t)\, dt \\
&= \int_{t_0}^{t_1} \bar{\bar{Z}}^{\mathsf{T}}(t) \cdot \bar{\bar{P}}_e(t) \cdot \bar{\bar{C}}^{\mathsf{T}}(t) \cdot \bar{\bar{R}}_e^{-1}(t) \cdot \bar{y}(t)\, dt
\end{aligned}$$

And,

$$\begin{aligned}
\frac{d}{dt_1}\bar{x}_e(t_1) &= \bar{\bar{Z}}^{\mathsf{T}}(t_1;t_1)\cdot\bar{\bar{P}}_e(t_1)\cdot\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{y}(t_1) \\
&+ \int_{t_0}^{t_1}\left(\frac{d}{dt_1}\bar{\bar{Z}}^{\mathsf{T}}(t;t_1)\right)\cdot\bar{\bar{P}}_e(t)\cdot\bar{\bar{C}}^{\mathsf{T}}(t)\cdot\bar{\bar{R}}_e^{-1}(t)\cdot\bar{y}(t)\,dt
\end{aligned}$$

Because

$$\begin{aligned}
\frac{d}{dt_1}\bar{\bar{Z}}(t;t_1) &= \frac{d}{dt_1}\bar{\bar{Z}}^{-1}(t_1;t) \\
&= -\bar{\bar{Z}}^{-1}(t_1;t)\cdot\left\{\Big[-\bar{\bar{A}}^{\mathsf{T}}(t_1)+\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{\bar{C}}(t_1)\cdot\bar{\bar{P}}_e(t_1)\Big]\cdot\bar{\bar{Z}}(t_1;t)\right\} \\
&\quad\cdot\bar{\bar{Z}}^{-1}(t_1;t) \\
&= -\bar{\bar{Z}}(t;t_1)\cdot\Big[-\bar{\bar{A}}^{\mathsf{T}}(t_1)+\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{\bar{C}}(t_1)\cdot\bar{\bar{P}}_e(t_1)\Big]
\end{aligned}$$

and

$$\frac{d}{dt_1}\bar{\bar{Z}}^{\mathsf{T}}(t;t_1) = \Big[\bar{\bar{A}}(t_1)-\bar{\bar{P}}_e(t_1)\cdot\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{\bar{C}}(t_1)\cdot\Big]\cdot\bar{\bar{Z}}^{\mathsf{T}}(t;t_1)$$

so,

$$\begin{aligned}
\frac{d}{dt_1}\bar{x}_e(t_1) &= \bar{\bar{P}}_e(t_1)\cdot\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{y}(t_1) \\
&+ \int_{t_0}^{t_1}\Big[\bar{\bar{A}}(t_1)-\bar{\bar{P}}_e(t_1)\cdot\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{\bar{C}}(t_1)\cdot\Big] \\
&\qquad\cdot\bar{\bar{Z}}^{\mathsf{T}}(t;t_1)\cdot\bar{\bar{P}}_e(t)\cdot\bar{\bar{C}}^{\mathsf{T}}(t)\cdot\bar{\bar{R}}_e^{-1}(t)\cdot\bar{y}(t)\,dt \\
&= \bar{\bar{P}}_e(t_1)\cdot\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{y}(t_1) \\
&+ \Big[\bar{\bar{A}}(t_1)-\bar{\bar{P}}_e(t_1)\cdot\bar{\bar{C}}^{\mathsf{T}}(t_1)\cdot\bar{\bar{R}}_e^{-1}(t_1)\cdot\bar{\bar{C}}(t_1)\Big]\cdot\bar{x}_e(t_1)
\end{aligned}$$

Hence, the optimal estimate $\bar{x}_e(t)$ of $\bar{x}(t)$ is defined by

$$\begin{aligned}
\frac{d}{dt}\bar{x}_e(t) &= \bar{\bar{A}}(t)\cdot\bar{x}_e(t)+\bar{\bar{L}}(t)\cdot\Big[\bar{\bar{C}}^{\mathsf{T}}(t)\cdot\bar{x}_e(t)-\bar{y}(t)\Big], \qquad \bar{x}_e(t_0)=0 \\
\text{where} &\qquad \bar{\bar{L}}(t)=\bar{\bar{P}}_e(t)\cdot\bar{\bar{C}}^{\mathsf{T}}(t)\cdot\bar{\bar{R}}_e^{-1}(t)
\end{aligned}$$

Adaptive Control. If the plant to be controlled is known exactly, it need design techniques to identify the parameters of physical processes and control them adaptively. Adaptive control is a technique of applying some system identification technique to obtain a model of the process and its environment from input-output experiments and using this method to design a controller. The parameters of the controller are adjusted during the operation of the plant as the amount of data available for plant identification increases.

One type of adaptive control scheme, called *Model Reference Adaptive Control*, is discussed.

The plant is described by:

$$\dot{\bar{x}}_p(t) = \bar{\bar{A}} \cdot \bar{x}_p(t) + \bar{\bar{Y}}(t) \cdot \bar{\Theta}^*$$

where $\bar{x}_p$ is the state of the plant, $\bar{\bar{A}}$ is a known stable system matrix, $\bar{\bar{Y}} \in R^{n \times r}$ is the matrix function of known state variables, and $\bar{\Theta}^* \in R^r$ is the vector of r unknown constant parameters.

The reference model is described by:

$$\dot{\bar{x}}_m(t) = \bar{\bar{A}} \cdot \bar{x}_m(t) + \bar{\bar{Y}}(t) \cdot \bar{\Theta}(t)$$

where $\bar{x}_m$ is the state of the reference model, $\bar{\Theta} \in R^r$ is the estimate of $\bar{\Theta}^*$.

Define

$$\begin{aligned} \bar{e}(t) &= \bar{x}_p(t) - \bar{x}_m(t) \\ \bar{\Phi}(t) &= \bar{\Theta}(t) - \bar{\Theta}^* \end{aligned}$$

and design the update law for the estimate of the unknown parameters as:

$$\dot{\bar{\Phi}}(t) = -\bar{\bar{\Gamma}} \cdot \bar{\bar{Y}}^{\mathsf{T}}(t) \cdot \bar{e}(t)$$

where $\bar{\bar{\Gamma}}$ is a weighting matrix and is assumed to be invertible. Therefore, it can be shown that the error $\bar{e}(t) = \bar{x}_p(t) - \bar{x}_m(t) \to 0$ as $t \to \infty$. Moreover, under additional conditions, $\bar{\Theta}(t) \to \bar{\Theta}^*$ as $t \to \infty$.

A Scalar Example of MRAC. The plant is described by [2]:

$$\dot{x}_p(t) = a_p \cdot x_p(t) + k_p \cdot u(t)$$

where x_p is the state of the plant, u_p is the input of the plant, and a_p, k_p are unknown constants.

The reference model is described by:

$$\dot{x}_m(t) = a_m \cdot x_m(t) + k_m \cdot r(t)$$

where x_m is the state of the reference model, r is the reference trajectory to both the plant and model, a_m, k_m are known parameters, specified by the designer.

Define:

$$\begin{aligned}\theta_1^* &= \frac{k_m}{k_p} \\ \theta_2^* &= \frac{a_m - a_p}{k_p},\end{aligned}$$

and an ideal controller is designed as follows:

$$u(t) = \theta_1^* \cdot r(t) + \theta_2^* \cdot x_p(t).$$

Since a_p, k_p are unknown, θ_1^*, θ_2^* cannot be computed directly. However, a set of update laws used to identify these two parameters can be designed as follows:

$$\begin{aligned}\dot{\theta}_1(t) &= -\gamma \cdot (x_p(t) - x_m(t)) \cdot r(t) \\ \dot{\theta}_2(t) &= -\gamma \cdot (x_p(t) - x_m(t)) \cdot x_p(t)\end{aligned}$$

where γ is any positive constant. Hence, the actual controller applied the plant is:

$$u(t) = \theta_1(t) \cdot r(t) + \theta_2(t) \cdot x_p(t)$$

The schematic diagram is shown in Figure 1.7. It can be further proved that the error between $x_p(t)$ and $x_m(t)$ will approach zero as $t \to \infty$. However, whether θ_1, θ_2 will aymptotically identify the true values of θ_1^*, θ_2^* cannot be guaranteed by the above design. If the reference trajectory $r(t)$ is persistently excting, the difference $(\theta_i(t) - \theta_i^*) \to 0$, as $t \to \infty$.

Robust Control. System parameters are unknown, but the bounds of the parameters are known.

For example,

$$\dot{\bar{x}}(t) = (\bar{\bar{A}} + \Delta\bar{\bar{A}}) \cdot \bar{x}(t) + \bar{\bar{B}} \cdot \bar{u}$$

where $\Delta\bar{\bar{A}}$ represents the uncertain part of the system matrix and its matrix norm is assumed to bounded by a constant number, i.e., $||\Delta\bar{\bar{A}}|| < \gamma$.

Therefore, when designing a state feedback controller, the uncertainty magnitude γ should be considered.

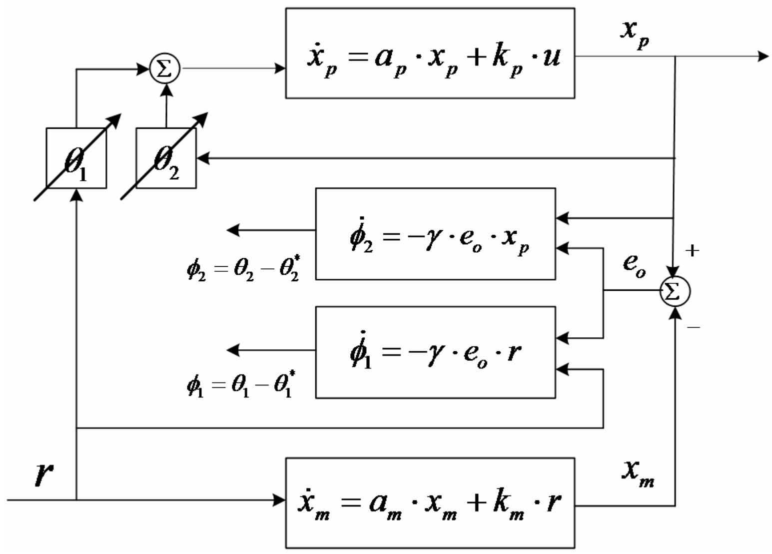


Figure 1.7. Example of model reference adaptive control.

For example, if a state feedback law is considered:

$$\bar{u}(t) \;=\; g(t, \gamma, \bar{r}, \bar{x}) \;=\; \bar{r} - \bar{\bar{K}} \cdot \bar{x}(t)$$

then, the state feedback gain matrix $\bar{\bar{K}}$ should be designed, such that the closed-loop system, including the uncertainty, is stable. That is, all the eigenvalues of $(\bar{\bar{A}} + \Delta\bar{\bar{A}} - \bar{\bar{B}} \cdot \bar{\bar{K}})$ have negative real part for all possible situation on $\Delta\bar{\bar{A}}$.

One example of robust control is shown in Figure 1.8, where Π is the plant dynamics with other uncertainties and K is the controller [7]. There are mainly three types of uncertainties.

The first one is the additive uncertainty, that is, $\Pi = P + \Delta$, as shown in Figure 1.9.

The second one is the multiplicative uncertainty which could be post- or pre-multicative, that is, $\Pi = P \cdot (I + \Delta)P$ or $\Pi = (I + \Delta) \cdot P$. as shown in Figures 1.11 and **??**, respectively.

The third one is the coprime factor uncertainty that is, $\Pi = (M + \Delta_M)^{-1} \cdot (N + \Delta_N)$ (left coprime factor), or $\Pi = (N + \Delta_N) \cdot (M + \Delta_M)^{-1}$ (right coprime factor) with $P = M^{-1} \cdot N$. Figure 1.12 shows the case of the left coprime factor uncertainty.

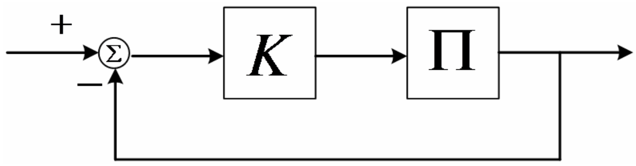


Figure 1.8. Example of robust control.

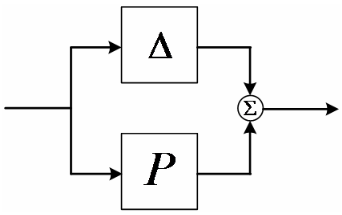


Figure 1.9. Example of robust control: Additive uncertainty, i.e., $\Pi = P + \Delta$.

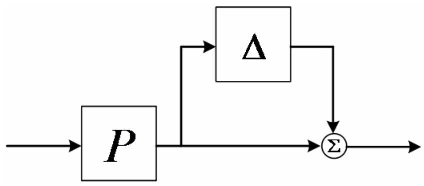


Figure 1.10. Example of robust control: Post-multiplicative uncertainty, i.e., $\Pi = P \cdot (I + \Delta)$.

References

[1] C.-T. Chen, *Linear System Theory and Design*, Third Edition, Oxford University Press, 1999.

[2] H. K. Khalil, *Nonlinear Systems*, Third Edition, Englewood Cliffs NJ: Prentice-Hall, 2002.

[3] S. Sastry, *Nonlinear Systems: Analysis, Stability, and Control*, Berlin: Springer-Verla, 1999.

[4] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Englewood Cliffs NJ: Prentice-Hall, 1990.

[5] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence, and Robustness*, Englewood Cliffs NJ: Prentice-Hall, 1989.

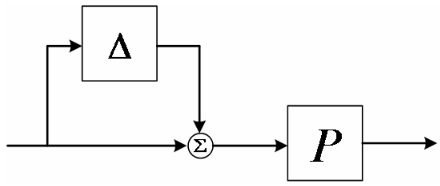


Figure 1.11. Example of robust control: Pre-multiplicative uncertainty, i.e., $\Pi = (I + \Delta) \cdot P$.

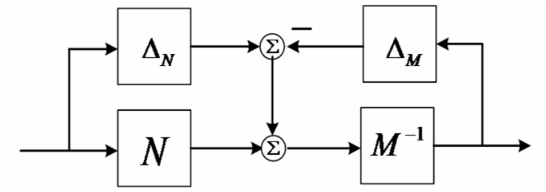


Figure 1.12. Example of robust control: Left coprime factor uncertainty, i.e., $\Pi = (M + \Delta_M)^{-1} \cdot (N + \Delta_N)$ with $P = M^{-1} \cdot N$.

[6] J. M. Maciejowski, *Multivariable Feedback Design*, Reading MA: Addison-Wesley, 1989.

[7] K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Upper Saddle River NJ: Prentice-Hall, 1998.