Solution of HW04 for Units 4A, 4B: Feedback Analysis, Control Systems

Assigned: October 22, 2021

Due: November 4, 2021 (23:59)

1. (U4A: Sensitivity)

4. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{s(s+a)}.$$

(a) Compute the sensitivity of the closed-loop transfer function to changes in the parameter A.

(b) Compute the sensitivity of the closed-loop transfer function to changes in the parameter a.

(c) If the unity gain in the feedback changes to a value of $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function with respect to β.

Solution:

(a)

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\dfrac{A}{s(s+a)}}{1+\dfrac{A}{s(s+a)}} = \frac{A}{s^2+as+A},$$

$$\frac{dT}{dA} = \frac{(s^2+as+A)-A}{(s^2+as+A)^2},$$

$$\mathcal{S}_A^T = \frac{A}{T}\frac{dT}{dA} = \frac{A(s^2+as+A)}{A}\frac{s^2+as}{(s^2+as+A)^2} = \frac{s(s+a)}{s(s+a)+A}.$$

(b)

$$\frac{dT}{da} = \frac{-sA}{(s^2+as+A)^2}.$$

$$\frac{a}{T}\frac{dT}{da} = \frac{a(s^2+as+A)}{A}\frac{-sA}{(s^2+as+A)^2}.$$

$$\mathcal{S}_a^T = \frac{-as}{s(s+a)+A}.$$

(c) In this case,

$$T(s) = \frac{G(s)}{1 + \beta G(s)},$$

$$\frac{dT}{d\beta} = \frac{-G(s)^2}{(1 + \beta G(s))^2},$$

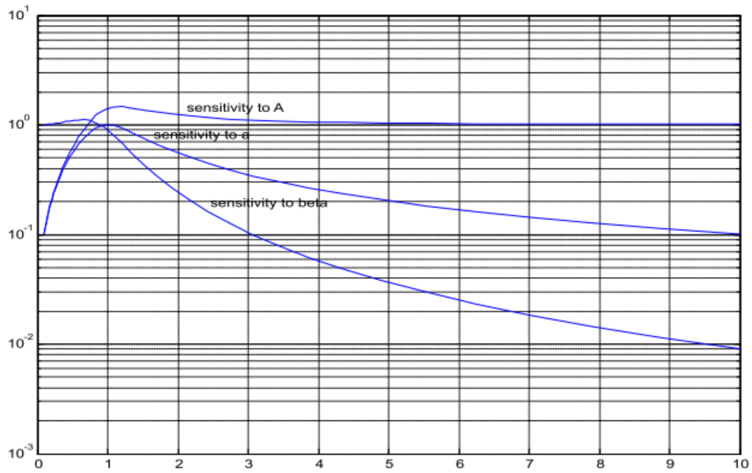
$$\frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1 + \beta G)}{G}\frac{-G^2}{(1 + \beta G)^2} = \frac{-\beta G}{1 + \beta G},$$

$$\mathcal{S}_\beta^T = \frac{\dfrac{-\beta A}{s(s + A)}}{1 + \dfrac{\beta A}{s(s + a)}} = \frac{-\beta A}{s(s + a) + \beta A}.$$

- If $a = A = 1$,the transfer function is most sensitive to variations in a and A near $\omega = 1$ rad/sec .
- The steady-state response is not affected by variations in A and a ($\mathcal{S}_A^T(0)$ and $\mathcal{S}_a^T(0)$ are both zero).
- The steady-state response *is* heavily dependent on β since $|\mathcal{S}_\beta^T(0)|$ $= 1.0$

See attached plots of sensitivities versus radian frequency for $a = A = 1.0$.

Sensitivity function frequency response follows.

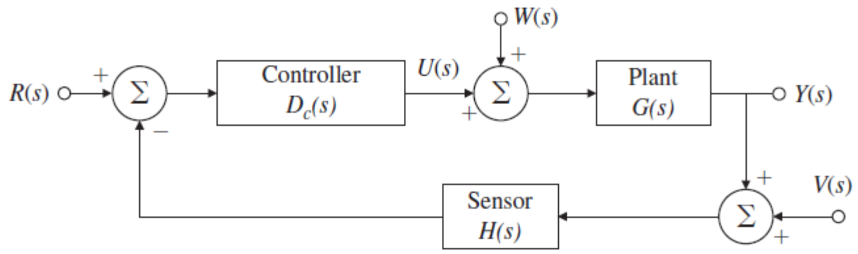


2. (U4B: Steady-State Error and System Type)

8. A standard feedback control block diagram is shown in Fig. 4.5 with

$$G(s) = \frac{1.5}{s};\ \ D_c(s) = \frac{(s+9)}{(s+3)}; H(s) = \frac{70}{(s+70)}; V(s) = 0.$$

(a) Let $W = 0$ and compute the transfer function from R to Y.
(b) Let $R = 0$ and compute the transfer function from W to Y.
(c) What is the tracking error if R a unit-step input and W = 0?
(d) What is the tracking error if R is a unit-ramp input and W = 0?
(e) What is the system type with respect to the reference inputs and the corresponding error coefficient?



Solution:

(a)

$$\frac{Y(s)}{R(s)} = T(s) = F(s)\frac{D_c(s)G(s)}{1 + D_c(s)G(s)H(s)} = \frac{1.5(s+9)(s+70)}{s^3 + 73s^2 + 315s + 945}$$

(b)

$$\frac{Y(s)}{W(s)} = T(s) = \frac{G(s)}{1 + D_c(s)G(s)H(s)} = \frac{1.5(s+3)(s+70)}{s^3 + 73s^2 + 315s + 945}$$

(c)

$$\frac{E(s)}{R(s)} = 1 - T(s) = S(s) = \frac{s^3 + 71.5s^2 + 196.5s}{s^3 + 73s^2 + 315s + 945}.$$

With $R(s) = \frac{1}{s}$,

$$e_{step}(\infty) = \lim_{s\to 0} sE(s) = 0.$$

(d)

With $R(s) = \frac{1}{s^2}$,

$$e_{ramp}(\infty) = \lim_{s\to 0} sE(s) = 0.2.$$

(e) System is Type 1 and K_v is

$$K_v = \frac{1}{|e_{ramp(\infty)}|} = 5\ sec^{-1}.$$

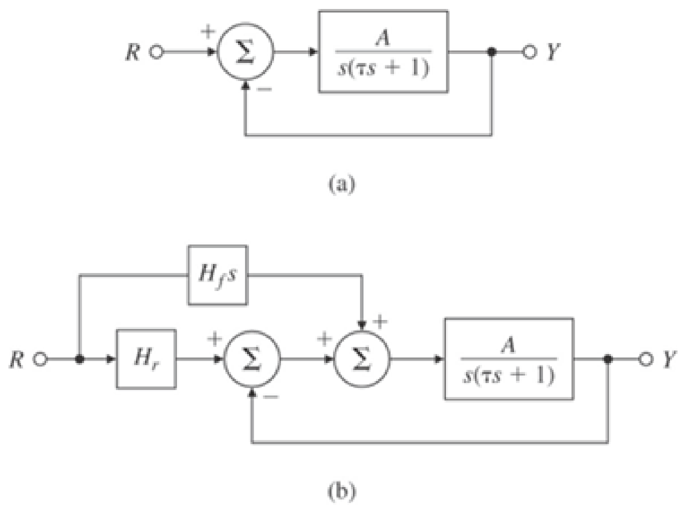
3. (U4B: System Type and Tracking)

14. Consider the system shown in Fig. 4.36(a).

    (a) What is the system type? Compute the steady-state tracking error due to a ramp input $r(t) = r_o t1(t)$.

    (b) For the modified system with a feed forward path shown in Fig.4.36(b), give the value of H_f so the system is Type 2 for reference inputs and compute the K_a in this case.

    (c) Is the resulting Type 2 property of this system robust with respect to changes in H_f i.e., will the system remain Type 2 if H_f changes slightly?



Solution:

(a) System is Type 1 since it is unity feedback and has a pole at $s = 0$in the forward path. Also,

$$E(s) = [1 - \mathcal{T}(s)]R(s),$$
$$= \left[\frac{1}{1+G(s)}\right] R(s),$$
$$= \frac{s(\tau s+1)}{s(\tau s+1)+A} \frac{r_o}{s^2}.$$

The steady-state tracking error using the FVT (assuming stability) is,

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{r_o}{A}.$$

(b)

$$Y(s) = \frac{A}{s(\tau s+1)} U(s),$$

$$U(s) = H_f s R(s) + H_r R(s) - Y(s),$$

$$Y(s) = \frac{A(H_f s + H_r)}{s(\tau s+1)+A} R(s).$$

The tracking error is,

$$E(s) = R(s) - Y(s),$$

$$\begin{aligned} &= \frac{s(\tau s+1)+A-A(H_f s+H_r)}{s(\tau s+1)+K} R(s), \\ &= \frac{\tau s^2+(1-AH_f)s+A(1-H_r)}{s(\tau s+1)+A}. \end{aligned}$$

To get zero steady-state error with respect to a ramp, the numerator in the above equation must have a factor s^2. For this to happen, let

$$\begin{aligned} H_r &= 1, \\ AH_f &= 1. \end{aligned}$$

Then

$$E(s) = \frac{\tau s^2}{s(\tau s+1)+A} R(s)$$

and, with $R(s) = \frac{r_o}{s^2}$, apply the FVT (assuming stability) to obtain

$$e_{ss} = 0.$$

Thus the system will be Type 2 with $K_a = \cancel{\frac{\tau}{A}}$. $\boxed{K_a = \frac{A}{\tau}}$

(c) No, the system is not robust Type 2 because the property is lost if either H_r or H_f changes slightly.