Solution of HW03 for Units 3D, 3E, 3F: Dynamic Response, Control Systems

Assigned: October 15, 2021

Due: October 28, 2021 (23:59)

1. (Effect of zeros and additional poles)

41. ▲ Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2+1}{(s/40+1)[(s/4)^2+s/4+1]}.$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find inverse Laplace transform.) Then compare your answer with the step response computed using MATLAB.

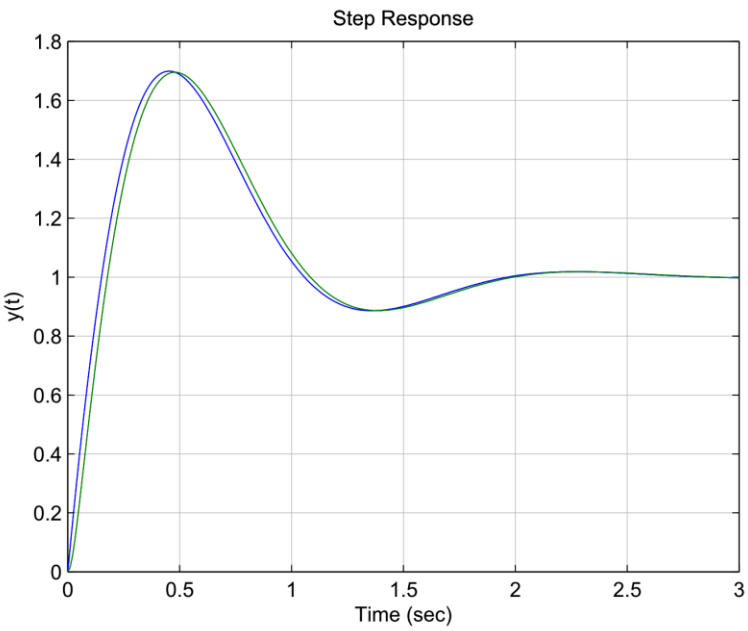
Solution:

From the location of the poles, we notice that the real pole is a factor of 20 away from the complex pair of poles. Therefore, the response of the system is *dominated* by the complex pair of poles.

$$G(s) \approx \frac{(s/2+1)}{[(s/4)^2+s/4+1]}.$$

This is now in the same form as equation (3.72) where $\alpha = 1$, $\zeta = 0.5$ and $\omega_n = 4$. Therefore, Fig. 3.24 suggests an overshoot of over 70%. The step response is the same as shown in Fig. 3.27, for $\alpha = 1$, with more than 70% overshoot and settling time of 3 seconds. The MATLAB plots below confirm this.

```
% Problem 3.41 FPE 8e

num=[1/2, 1];

den1=[1/16, 1/4, 1];

sys1=tf(num,den1);

t=0:.01:3;

y1=step(sys1,t);

den=conv([1/40, 1],den1);

sys=tf(num,den);

y=step(sys,t);

plot(t,y1,t,y);

xlabel('Time (sec)');

ylabel('y(t)');

title('Step Response');

grid on;
```



Problem 3.41: Comparison of step responses: third-order system (green), second-order approximation (blue).

2. (Effect of zeros and additional poles)

46. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}, \tag{1}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \tag{2}$$

(a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an *odd* number of *real* RHP zeros.

Solution:

(a) For $G_1(s)$:

$$\begin{aligned} Y_1(s) &= \frac{1}{s}G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)}, \\ H(s) &= k\frac{\prod^j (s-z_j)}{\prod^l (s-p_l)}, \\ R_{p_i} &= \lim_{s\to p_i}[(s-p_i)H(s)] = \lim_{s\to p_i} k\frac{\prod^j (s-z_j)}{\prod^l_{l\neq i}(s-p_l)} = k\frac{\prod^j (p_i-z_j)}{\prod^l_{l\neq i}(p_i-p_l)}. \end{aligned}$$

Each factor $(p_i - z_j)$ or $(p_i - p_l)$ can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to p_i and coming from z_j or p_l respectively.

The method for calculating the residue at a pole p_i is:

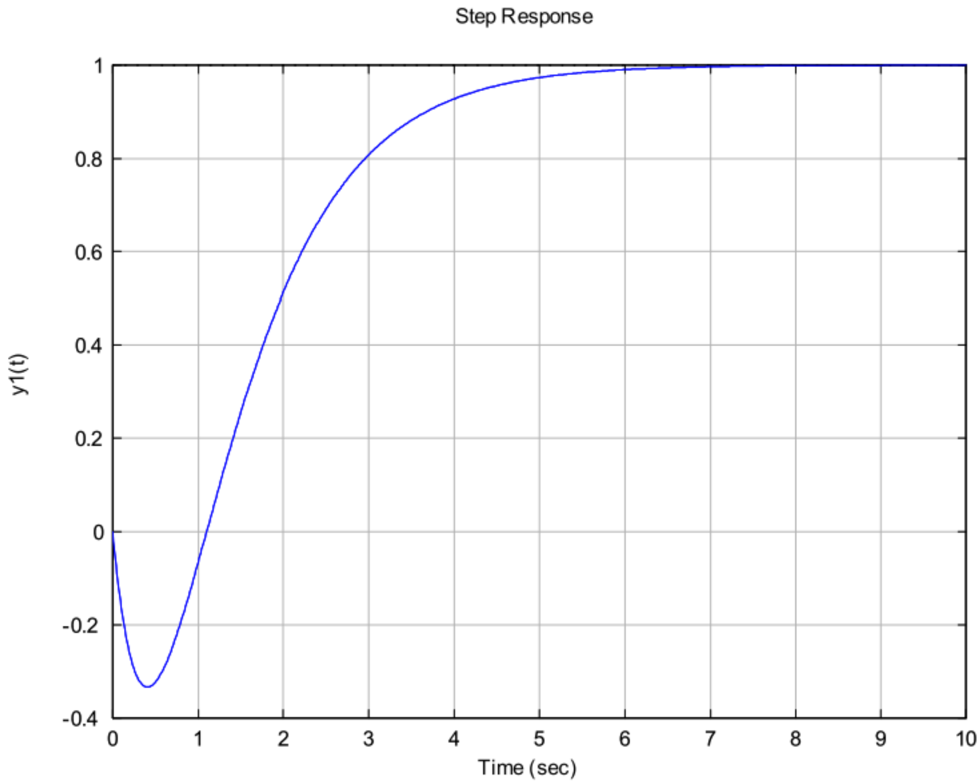
(1) Draw vectors from the rest of the poles and from all the zeros to the pole p_i.

(2) Measure magnitude and phase of these vectors.

(3) The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.

In our problem:

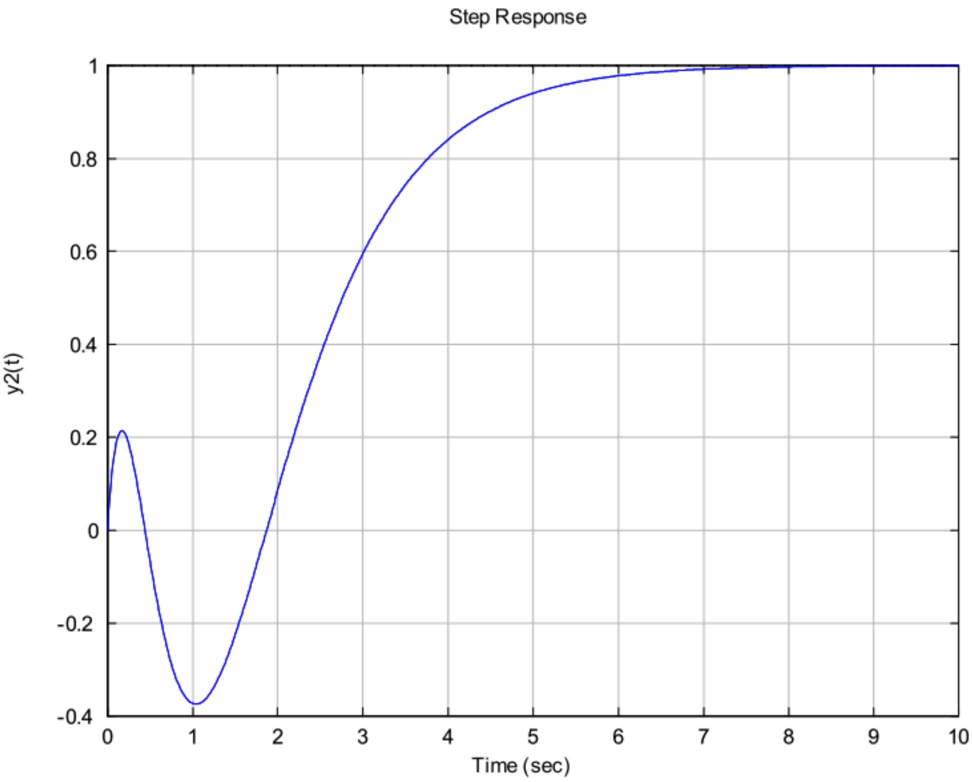
$$
\begin{aligned}
Y_1(s) &= \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s+1)} + \frac{R_{-2}}{(s+2)} = \frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2}, \\
y_1(t) &= 1 - 4e^{-t} + 3e^{-2t}.
\end{aligned}
$$



Problem 3.47: Step response for a non-minimum phase system with one *real* RHP zero.

For $G_2(s)$:

$$
\begin{aligned}
Y_2(s) &= \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{(s+1)} + \frac{18}{(s+2)} + \frac{-10}{(s+3)}, \\
y_2(t) &= 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t}.
\end{aligned}
$$



Problem 3.47: Step response of a non-minimum phase system with two *real* zeros in the RHP.

(b) The first system presents an "undershoot". The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In $y_1(t)$: dominant at $t = 0$ the term $-4e^{-t}$

In $y_2(t)$: dominant at $t = 0$ the term $18e^{-2t}$

(c) The following concise proof is from Reference [1] (see also References [2]-[3]).

Without loss of generality assume the system has unity DC gain $(G(0) = 1)$ Since the system is stable, $y(\infty) = G(0) = 1$, and it is reasonable to assume $y(\infty) \neq 0$. Let us denote the pole-zero excess as $r = n - m$. Then, $y(t)$ and its $r - 1$ derivatives are zero at $t = 0$, and $y^r(0)$ is the first non-zero derivative. The system has an undershoot if $y^r(0)y(\infty) < 0$. The transfer function may be re-written as

$$G(s) = \frac{\prod_{i=1}^{m}(1 - \frac{s}{z_i})}{\prod_{i=1}^{m+r}(1 - \frac{s}{p_i})}$$

The *numerator* terms can be classified into three types of terms:
(1). The first group of terms are of the form $(1 - \alpha_i s)$ with $\alpha_i > 0$.
(2). The second group of terms are of the form $(1 + \alpha_i s)$ with $\alpha_i > 0$.
(3). Finally, the third group of terms are of the form, $(1 + \beta_i s + \alpha_i s^2)$ with $\alpha_i > 0$, and β_i could be negative.
However, $\beta_i^2 < 4\alpha_i$, so that the corresponding zeros are complex.
All the *denominator* terms are of the form (2), (3), above. Since,

$$y^r(0) = \lim_{s\to\infty} s^r G(s)$$

it is seen that the *sign* of $y^r(0)$ is determined entirely by the number of terms of group 3 above. In particular, if the number is *odd*, then $y^r(0)$ is *negative* and if it is even, then $y^r(0)$ is positive. Since $y(\infty) = G(0) = 1$, then we have the desired result.

References

[1] Vidyasagar, M., "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automat. Contr.*, Vol. AC-31, p. 440, May 1986.

[2] Clark, R., N., *Introduction to Automatic Control Systems*, John Wiley, 1962.

[3] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms, " *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 402-407, 1981.

3. Stability)

51. A measure of the degree of instability in an unstable aircraft response is the amount of time it takes for the *amplitude* of the time response to double (see Fig. 3.65), given some nonzero initial condition.

    (a) For a first-order system, show that the **time to double** is

    $$\tau_2 = \frac{\ln 2}{p},$$

    where p is the pole location in the RHP.

    (b) For a second-order system (with two complex poles in the RHP), show that

    $$\tau_2 = \frac{\ln 2}{-\zeta\omega_n}.$$

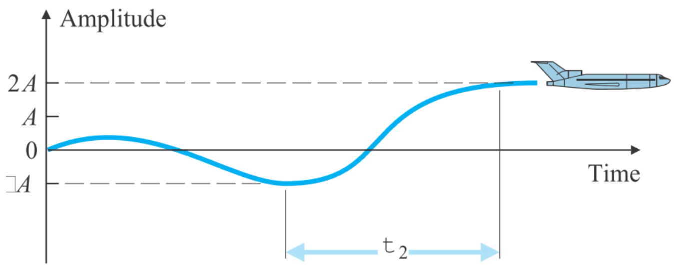


Figure 3.65: Time to double

Solution:

(a) First-order system, $H(s)$ could be:

$$\begin{aligned}
H(s) &= \frac{k}{(s-p)}, \\
h(t) &= \mathcal{L}^{-1}\left[H(s)\right] = ke^{pt}, \\
h(\tau_0) &= ke^{p\tau_0}, \\
h\left(\tau_0+\tau_2\right) &= 2h\left(\tau_0\right) = ke^{p(\tau_0+\tau_2)},
\end{aligned}$$

$$\Longrightarrow \quad 2ke^{p\tau_0} = ke^{p\tau_0}e^{p\tau_2},$$

$$\Longrightarrow \quad \tau_2 = \frac{\ln 2}{p}.$$

$$
\begin{aligned}
|t_0| &= -y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1-|\zeta^2|}}, \\
|\tau_0| &= -y_0 \frac{e^{\omega_n |\zeta| \tau_0}}{\sqrt{1-|\zeta^2|}}, \\
|\tau_0+\tau_2| &= -y_0 \frac{e^{\omega_n |\zeta| (\tau_0+\tau_2)}}{\sqrt{1-|\zeta^2|}} = 2\,|\tau_0|
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \qquad & e^{\omega_n |\zeta| \tau_2} = 2 \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{\omega_n\,|\zeta|} = \frac{\ln 2}{-\omega_n \zeta} \qquad (\zeta \leq 0)
\end{aligned}
$$

Note: This problem shows that $\sigma = \omega_n\,|\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

(b) Second-order system:

$$
y(t) = y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1-|\zeta^2|}} \sin\left(\omega_n \sqrt{1-|\zeta^2|}\,t + \cos^{-1}\zeta\right),
$$

where

$$
\cos^{-1}\zeta = \cos^{-1}|\zeta| + \pi
$$

$$
\Longrightarrow y(t) = y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1-|\zeta^2|}} (-1) \sin\left(\omega_n \sqrt{1-|\zeta^2|}\,t + \cos^{-1}|\zeta|\right)
$$

Note: Instead of working with a negative ζ, everything is changed to $|\zeta|$.

$$
\begin{aligned}
|t_0| &= -y_0 \frac{e^{\omega_n|\zeta|t}}{\sqrt{1-|\zeta^2|}}, \\
|\tau_0| &= -y_0 \frac{e^{\omega_n|\zeta|\tau_0}}{\sqrt{1-|\zeta^2|}}, \\
|\tau_0+\tau_2| &= -y_0 \frac{e^{\omega_n|\zeta|(\tau_0+\tau_2)}}{\sqrt{1-|\zeta^2|}} = 2|\tau_0|
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \qquad & e^{\omega_n|\zeta|\tau_2} = 2 \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{\omega_n|\zeta|} = \frac{\ln 2}{-\omega_n\zeta} \qquad (\zeta \leq 0)
\end{aligned}
$$

Note: This problem shows that $\sigma = \omega_n|\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

4. (Stability)

55. The transfer function of a typical tape-drive system is given by

$$KG(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + KG(s) = 0$.

Solution:

$$1 + KG(s) = s^5 + 1.9s^4 + 5.1s^3 + 6.2s^2 + (2+K)s + 4K = 0.$$

The Routh array is,

$$\begin{array}{lclll}
s^5 & : & 1.0 & 5.1 & 2+K \\
s^4 & : & 1.9 & 6.2 & 4K \\
s^3 & : & a_1 & a_2 & \\
s^2 & : & b_1 & 4K & \\
s^1 & : & c_1 & & \\
s^0 & : & 4K & &
\end{array}$$

where

$$\begin{aligned}
a_1 &= \frac{(1.9)(5.1)-(1)(6.2)}{1.9} = 1.837 \qquad a_2 = \frac{(1.9)(2+K)-(1)(4K)}{1.9} = 2-1.1K \\
b_1 &= \frac{(a_1)(6.2)-(a_2)(1.9)}{a_1} = 1.138(K+3.63) \\
c_1 &= \frac{(b_1)(a_2)-(4K)(a_1)}{b_1} = \frac{-(1.25K^2+9.61K-8.26)}{1.138(K+363)} = \frac{-(K+8.47)(K-0.78)}{0.91(K+3.63)}
\end{aligned}$$

For stability we must have all the elements in the first column of the routh array to be positive, and that results in the following set of constraints:

$$\begin{array}{llcl}
(1) & b_1 & = & K+3.63 > 0 \Longrightarrow K > -3.63, \\
(2) & c_1 & > & 0 \Longrightarrow -8.43 < K < 0.78, \\
(3) & d_1 & > & 0 \Longrightarrow K > 0.
\end{array}$$

Intersection of (1), (2), and (3) $\Longrightarrow 0 < K < 0.78$.