Solution of HW02 for Units 3A, 3B: Dynamic Response, Control Systems

Assigned: October 8, 2021

Due: October 21, 2021 (23:59)

1. (Laplace Transform)

3.12 Consider the standard second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

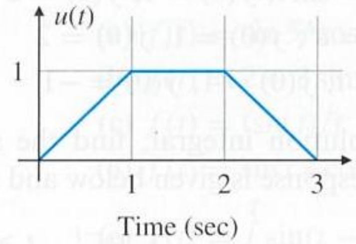
(a) Write the Laplace transform of the signal in Fig. 3.47.

(b) What is the transform of the output if this signal is applied to $G(s)$?

(c) Find the output of the system for the input shown in Fig. 3.47.

Figure 3.47

Plot of input signal for Problem 3.12



Solution:

(a) The input signal may be written as:

$$u(t) = t - (t-1) * 1(t-1) - (t-2) * 1(t-2) + (t-3) * 1(t-3),$$

where $1(t-\tau)$ denotes a delayed unit step. The Laplace transform of the input signal is:

$$U(s) = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

We can verify this in MATLAB:

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>> ilaplace(1/s^2*(1-exp(-s)-exp(-2*s)+exp(-3*s)))

ans =

t-heaviside(t-1)*(t-1)-heaviside(t-2)*(t-2)+heaviside(t-3)*(t-3)
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(b) The Laplace transform of the output if this input signal is applied is:

$$Y(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(\left(\frac{1}{s^2}\right))(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

(c) However to make the mathematical manipulation easier, consider only the response of the system to a (unit) ramp input:

$$Y_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(\left(\frac{1}{s^2}\right)).$$

Partial fractions yields the following:

$$Y_1(s) = \frac{1}{s^2} - \frac{\frac{2\zeta}{\omega_n}}{s} + \frac{\frac{2\zeta}{\omega_n}(s + 2\zeta\omega_n - \frac{\omega_n}{2\zeta})}{(s + \omega_n\zeta)^2 + (\omega_n\sqrt{1 - \zeta^2})^2}.$$

Use the following Laplace transform pairs for the case $0 \leq \zeta < 1$:

$$L^{-1}\{\left\{\frac{s + z_1}{(s + a)^2 + \omega^2}\right\}\} = \sqrt{\frac{(z_1 - a)^2 + \omega^2}{\omega^2}} e^{-at} \sin(\omega t + \phi),$$

where

$$\phi \equiv \tan^{-1}(\left(\frac{\omega}{z_1 - a}\right)).$$

$$L^{-1}\{\left\{\frac{1}{s^2}\right\}\} = t \qquad \text{unit ramp}$$

$$L^{-1}\{\left\{\frac{1}{s}\right\}\} = 1(t) \qquad \text{unit step}$$

and the following Laplace transform pairs for the case $\zeta = 1$:

$$L^{-1}\{\left\{\frac{1}{(s + a)^2}\right\}\} = te^{-at}.$$

$$L^{-1}\{\left\{\frac{s}{(s + a)^2}\right\}\} = (1 - at)e^{-at}.$$

$$L^{-1}\{\left\{\frac{1}{s^2}\right\}\} = t \qquad \text{unit ramp,}$$

$$L^{-1}\{\left\{\frac{1}{s}\right\}\} = 1(t) \qquad \text{unit step,}$$

the following is derived:

$$y_1(t) = \{ \begin{array}{ll} t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}} \sin(\omega_n\sqrt{1 - \zeta^2}t + \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}) & \begin{array}{c} 0 \leq \zeta < 1 \\ t \geq 0 \end{array} \\ t - \frac{2}{\omega_n} + \frac{2}{\omega_n}e^{-\omega_n t}(\frac{\omega_n}{2}t + 1) & \begin{array}{c} \zeta = 1 \\ t \geq 0 \end{array} \end{array}.$$

Since $u(t)$ consists of a ramp and three delayed ramp signals, using superposition (the system is linear), then:

$$y(t) = y_1(t) - y_1(t-1) - y_1(t-2) + y_1(t-3) \qquad t \geq 0.$$

2. (Laplace Transform)

3.16 For a second-order system with transfer function

$$G(s) = \frac{5}{s^2 + s + 4},$$

Determine the following:

(a) The DC gain and whether the system is stable.

(b) The final value of the output if the input is applied with a step of 2 units or $R(s) = \frac{2}{s}$.

Solution:

(a) The system is stable and therefore the Final Value Theorem is applicable, and the DC gain is

$$G(0) = \frac{5}{4} = 1.25$$

By finding the roots of $s^2 + s + 4 = 0$, the poles are located at $s = -0.5 \pm j1.94$. Since the poles are at the LHS of s-plane, the system is stable.

(b) With a step input of 2 units or $R(s) = \frac{2}{s}$, the final value can be calculated as

$$\lim_{s \to 0} sG(s)R(s) = 2G(0) = 2.5 \text{ units.}$$

3. (Block diagram)

3.20 Find the transfer functions for the block diagrams in Fig. 3.50.

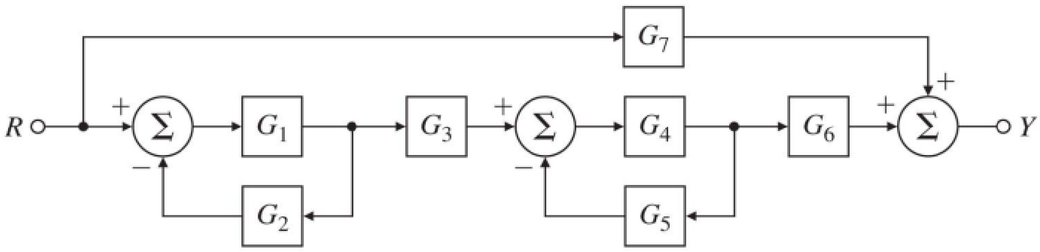
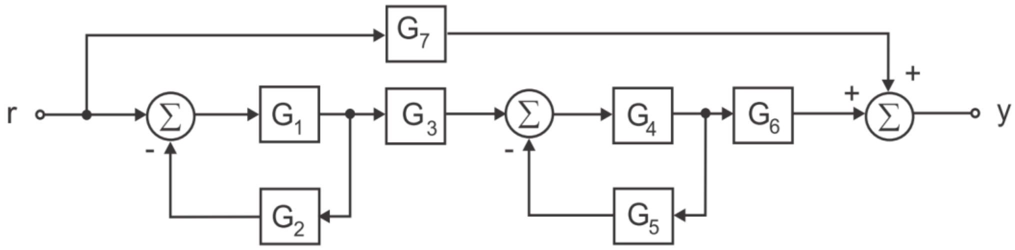
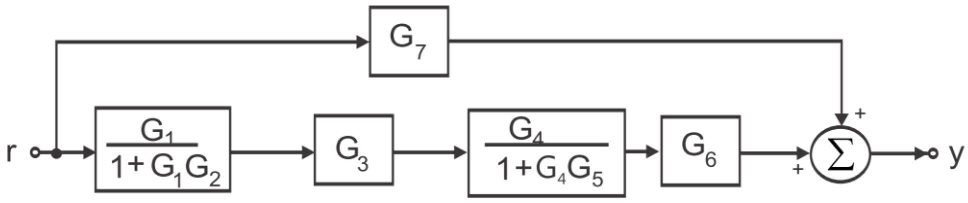


Figure 3.50 Block diagrams for Problem 3.20

Solution:

(b)





$$\frac{Y}{R} = G_7 + \frac{G_1G_3G_4G_6}{(1+G_1G_2)(1+G_4G_5)}.$$

4. (Time domain specification)

26. For the unity feedback system shown in Fig. 3.55, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 sec. Verify your design using MATLAB.

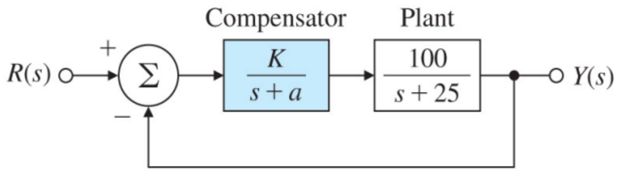


Figure 3.55: Unity feedback system for Problem 3.26

Solution:

$$\frac{Y(s)}{R(s)} = \frac{100K}{s^2 + (25+a)s + 25a + 100K} = \frac{100K}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Using the given information:

$$\begin{aligned} R(s) &= \frac{1}{s} \qquad \text{unit step,} \\ M_p &\leq 25\%, \\ t_s &\leq 0.1\,\text{sec}\,. \end{aligned}$$

Solve for ζ :

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \\ \zeta &= \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.4037. \end{aligned}$$

Solve for ω_n:

$$e^{-\zeta\omega_n t_s} = 0.01 \qquad \text{For a 1\% settling time.}$$

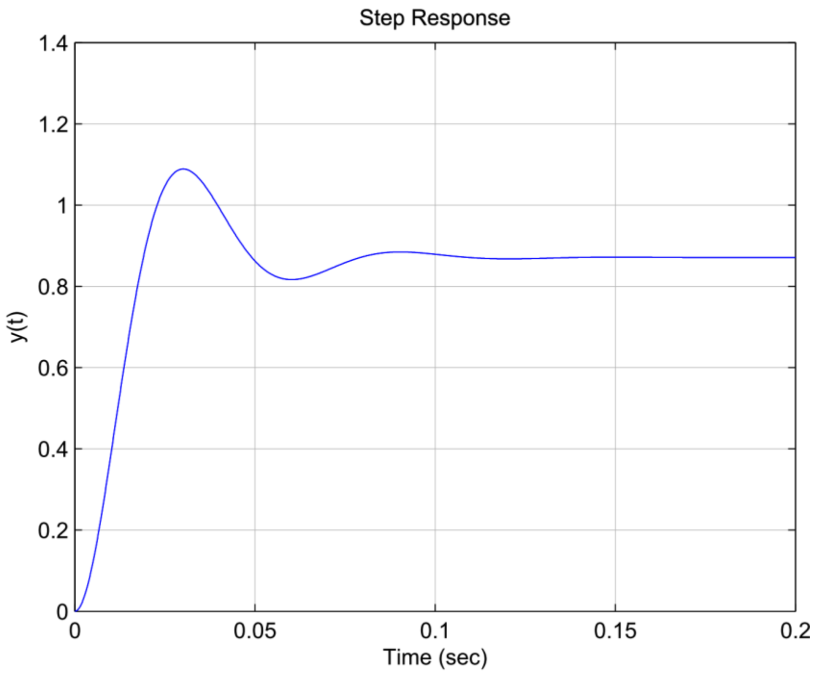
$$t_s \leq \frac{4.605}{\zeta\omega_n} = 0.1,$$

$$\Longrightarrow \omega_n \approx 114.07.$$

Now find a and K :

$$
\begin{aligned}
2\zeta\omega_n &= (25 + a), \\
a = 2\zeta\omega_n - 25 &= 92.10 - 25 = 67.10, \\
\omega_n^2 &= (25a + 100K), \\
K = \frac{\omega_n^2 - 25a}{100} &\approx 113.34.
\end{aligned}
$$

The step response of the system using MATLAB is shown next.



Step response for Problem 3.26.