

Fall 2021 (110-1)

控制系統  
Control Systems

Unit 6A  
Frequency Response

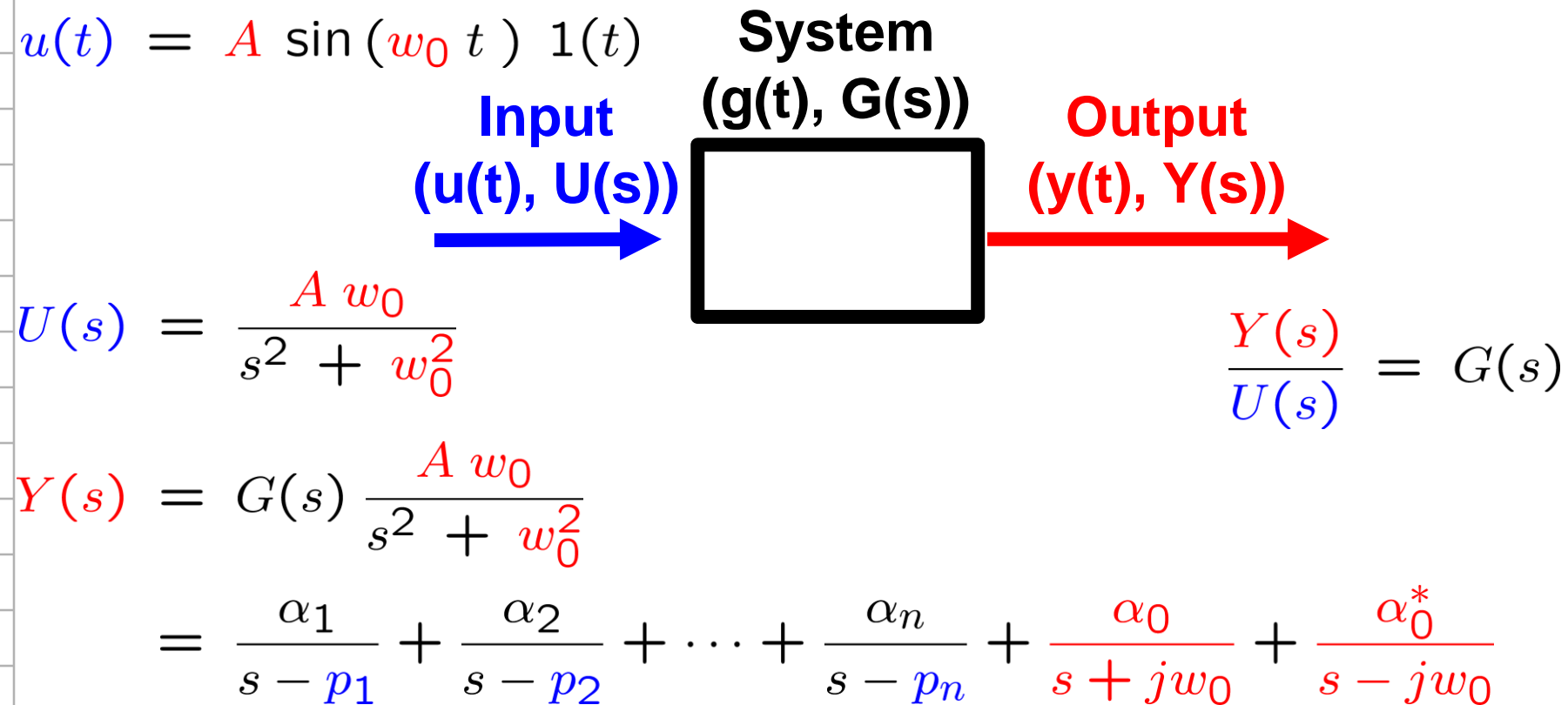
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## ■ The System's Frequency Response:

- A linear system's response to sinusoidal inputs
- Can be obtained from the knowledge of its pole and zero locations.

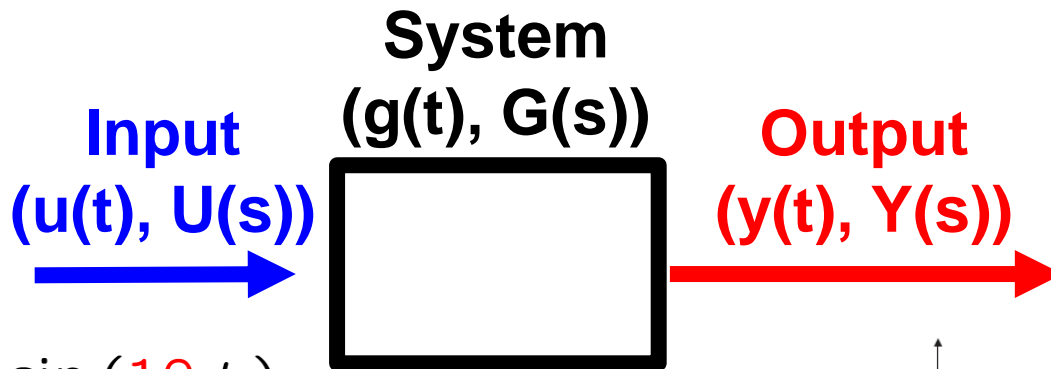


$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \dots + \alpha_n e^{p_n t} + 2 |\alpha_0| \cos(\omega_0 t + \phi)$$

for  $t \geq 0$

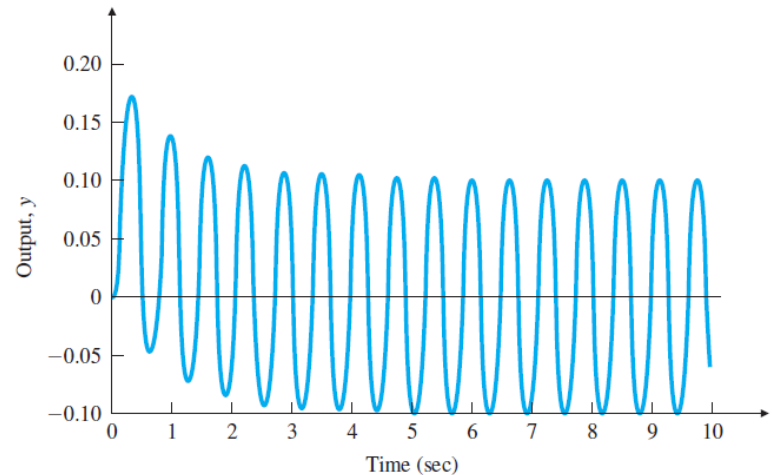
$$\phi = \tan^{-1} \left[ \frac{\text{Im}(\alpha_0)}{\text{Re}(\alpha_0)} \right]$$

- If all the **poles** of the system represent **stable behavior** (the **real** parts of  $p_1, p_2, \dots, p_n < 0$ ),
- the **natural unforced response** will **die out** eventually,
- and therefore the **steady-state response** of the system will be due solely to **the sinusoidal term**.



$$u(t) = \sin(10t)$$

$$G(s) = \frac{1}{s+1}$$

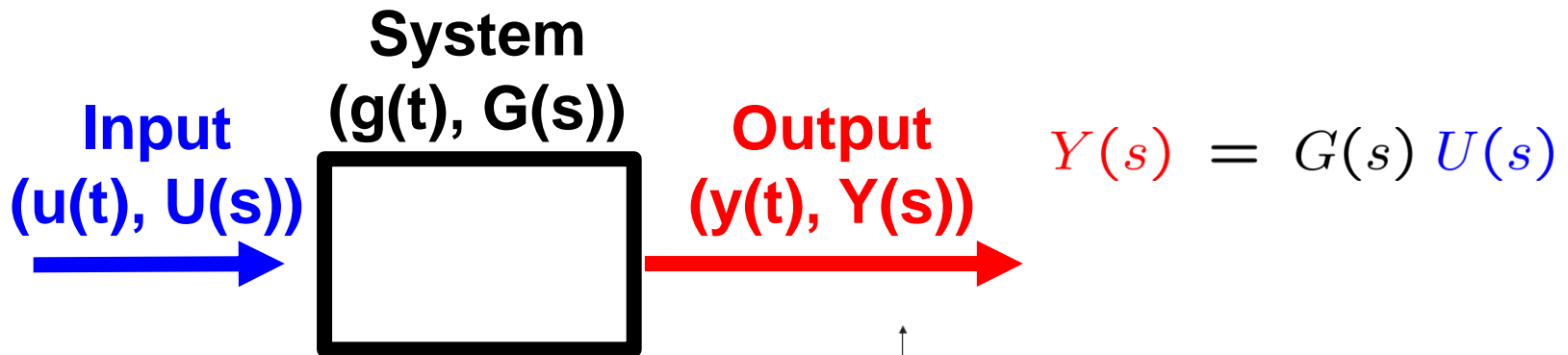


$$Y(s) = \frac{1}{s+1} \frac{10}{s^2 + 10^2}$$

$$= \frac{10}{101} \frac{1}{s+1} + \frac{2(1-j10)}{s+j10} + \frac{-j}{2(1+j10)}$$

$$y(t) = \frac{10}{101} e^{-t} + \frac{1}{\sqrt{101}} \sin(10t + \phi)$$

$$\begin{aligned} \phi &= \tan^{-1}(-10) \\ &= -84.2^\circ \end{aligned}$$



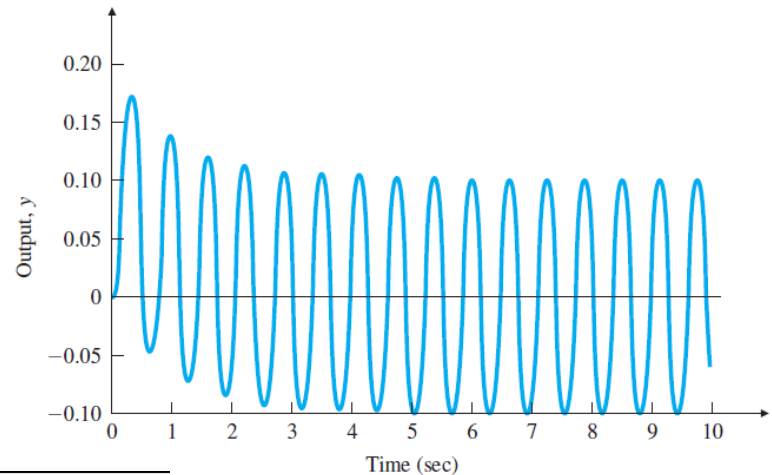
$$u(t) = A \sin(\omega_0 t)$$

$$y(t) \rightarrow A M \sin(\omega_0 t + \phi)$$

$$M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0}$$

$$= \sqrt{\{\text{Re}[G(j\omega_0)]\}^2 + \{\text{Im}[G(j\omega_0)]\}^2}$$

$$\phi = \tan^{-1} \left[ \frac{\text{Im}[G(j\omega_0)]}{\text{Re}[G(j\omega_0)]} \right] = \angle G(j\omega_0)$$



$$G(j\omega_0) = M e^{j\phi} = M(\omega_0) e^{j\phi(\omega_0)}$$

- Magnitude
- Phase

## ▪ Example 6.1:

## Frequency-Response Characteristics of a Capacitor

$$i = C \frac{dv}{dt}$$

$$G(s) = \frac{I(s)}{V(s)} = C s$$

$$G(j\omega) = C j \omega$$

$$M = |C j \omega| = C \omega$$

$$\phi = \angle G(C j \omega) = 90^\circ$$

▪ Magnitude

▪ Phase

- Example 6.2:  
Frequency-Response Characteristics of a Lead Compensator

$$D_c(s) = K \frac{T s + 1}{\alpha T s + 1}, \quad \alpha < 1$$

$$D_c(s) = K \frac{s + z}{s + p}$$

$z < p$

$$D_c(j\omega) = K \frac{T(j\omega) + 1}{\alpha T(j\omega) + 1}$$

- Frequency: Low vs High

$$M = |D_c| = |K| \frac{\sqrt{1 + (T\omega)^2}}{\sqrt{1 + (\alpha T\omega)^2}} \quad |K| \quad |K/\alpha|$$

$$\phi = \angle(1 + j\omega T) - \angle(1 + j\alpha\omega T) \quad 0 \quad + \quad 0$$

$$= \tan^{-1}[\omega T] - \tan^{-1}[\alpha\omega T]$$

# Examples

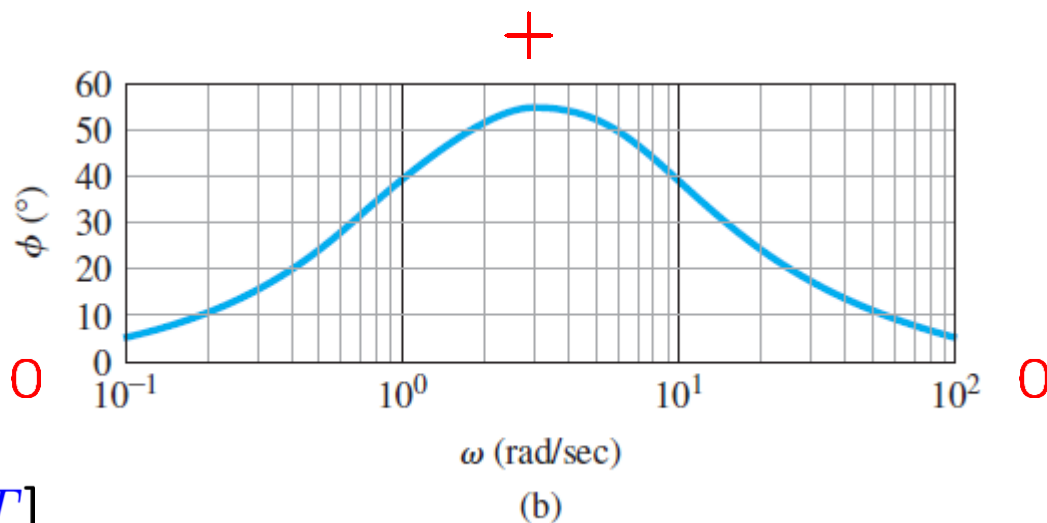
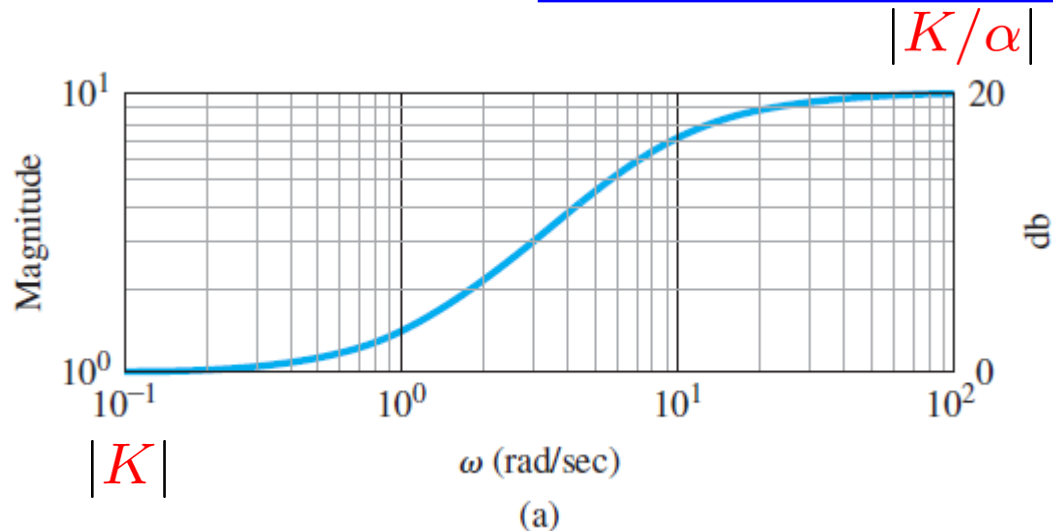
## Example 6.2: Frequency-Response Characteristics of a Lead Compensator

$\alpha = 1/10$

```
sysD = (s+1)/(s/10+1);
w = logspace(-1,2);
[mag,ph] = bode( sysD, w );
loglog( w, squeeze(mag) );
```

$$|K| \frac{\sqrt{1 + (Tw)^2}}{\sqrt{1 + (\alpha Tw)^2}}$$

$$\tan^{-1} [wT] - \tan^{-1} [\alpha wT]$$

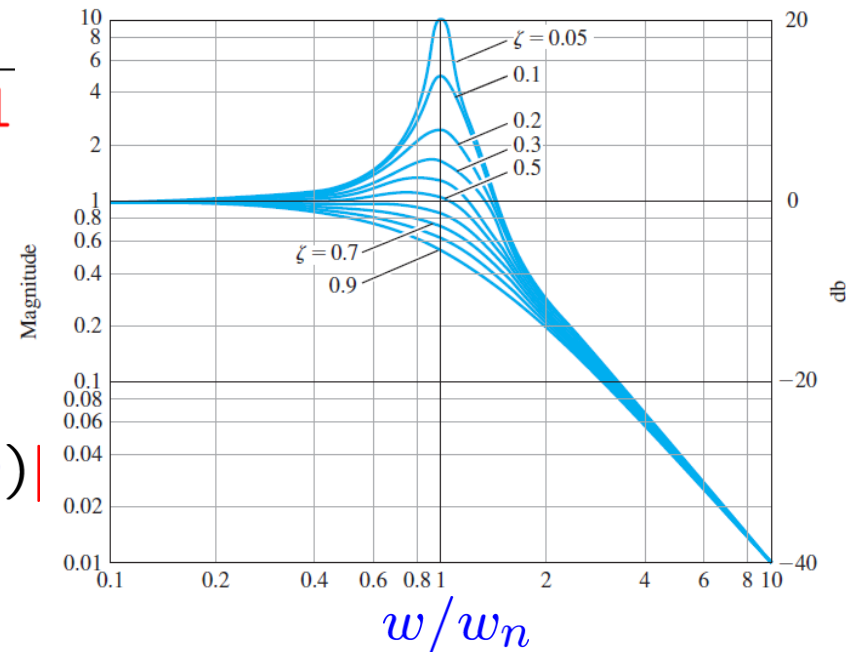




$$G(s) = \frac{1}{(s/w_n)^2 + 2\zeta(s/w_n) + 1}$$

Peak Overshoot  $\approx \frac{1}{2\zeta}$ ,  
For  $\zeta < 0.5$

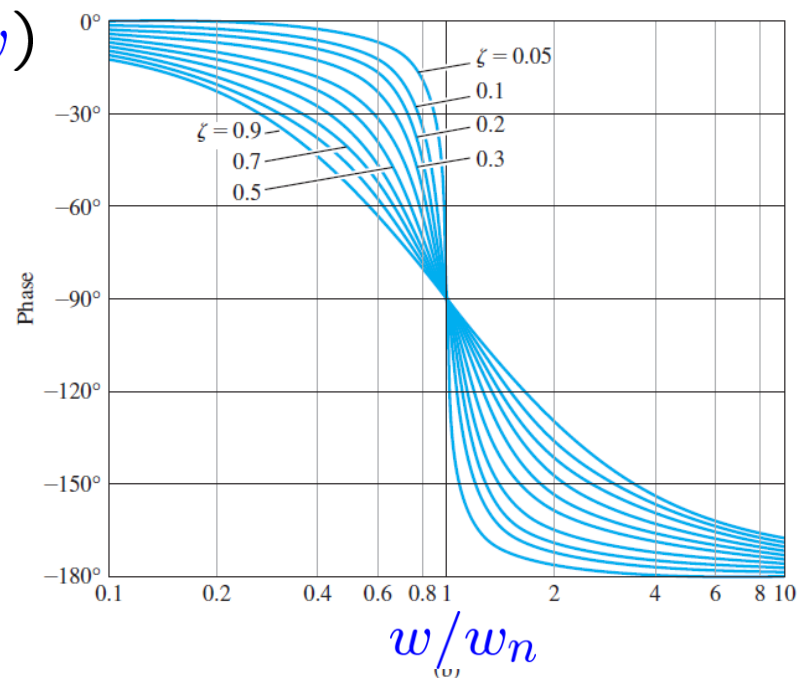
$$|G(s = jw)|$$

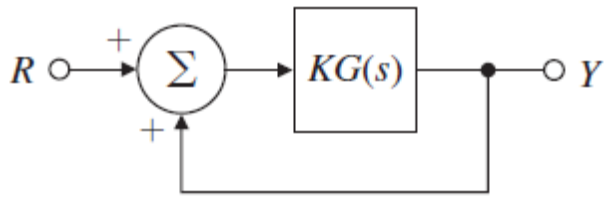


```
w = logspace(-1,1,200);    ∠G(s = jw)
zeta = 0.05;
% set_zeta: 0.05, 0.1, ..., 0.9;
```

```
numG = 1;
denG = [1 2*zeta 1];
[m, p] = bode(numG, denG, w);
```

```
loglog(w, m);
semilogx(w, p);
```



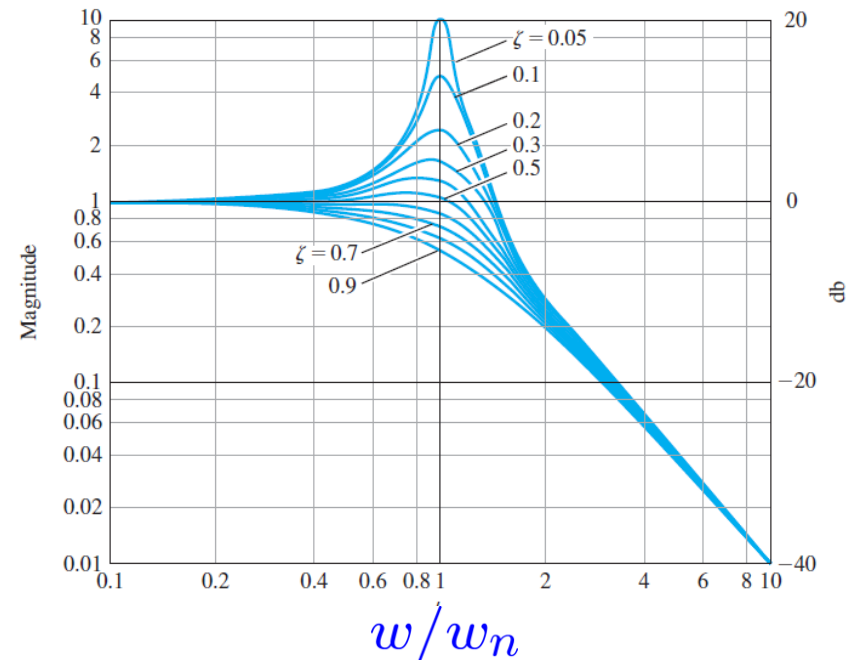
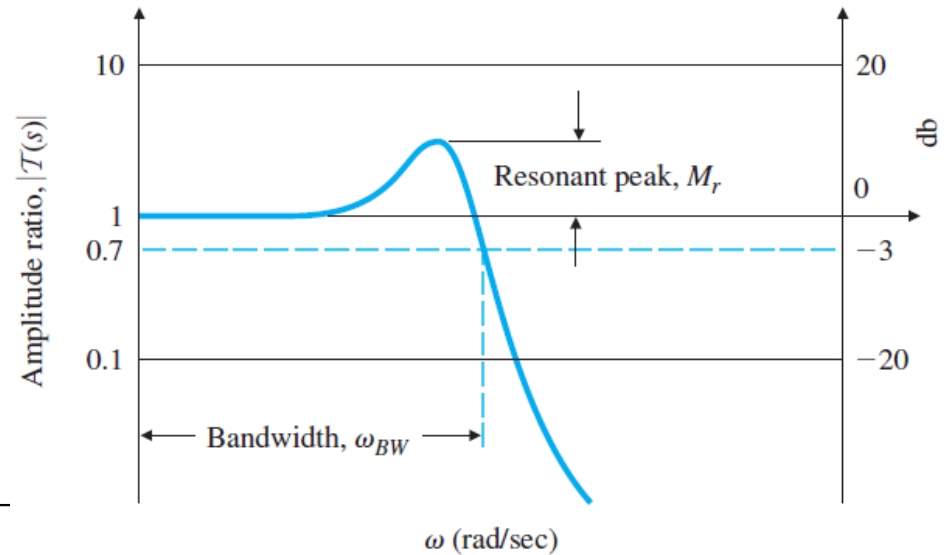


$$T(s) = \frac{Y(s)}{R(s)} = \frac{K G(s)}{1 + K G(s)}$$

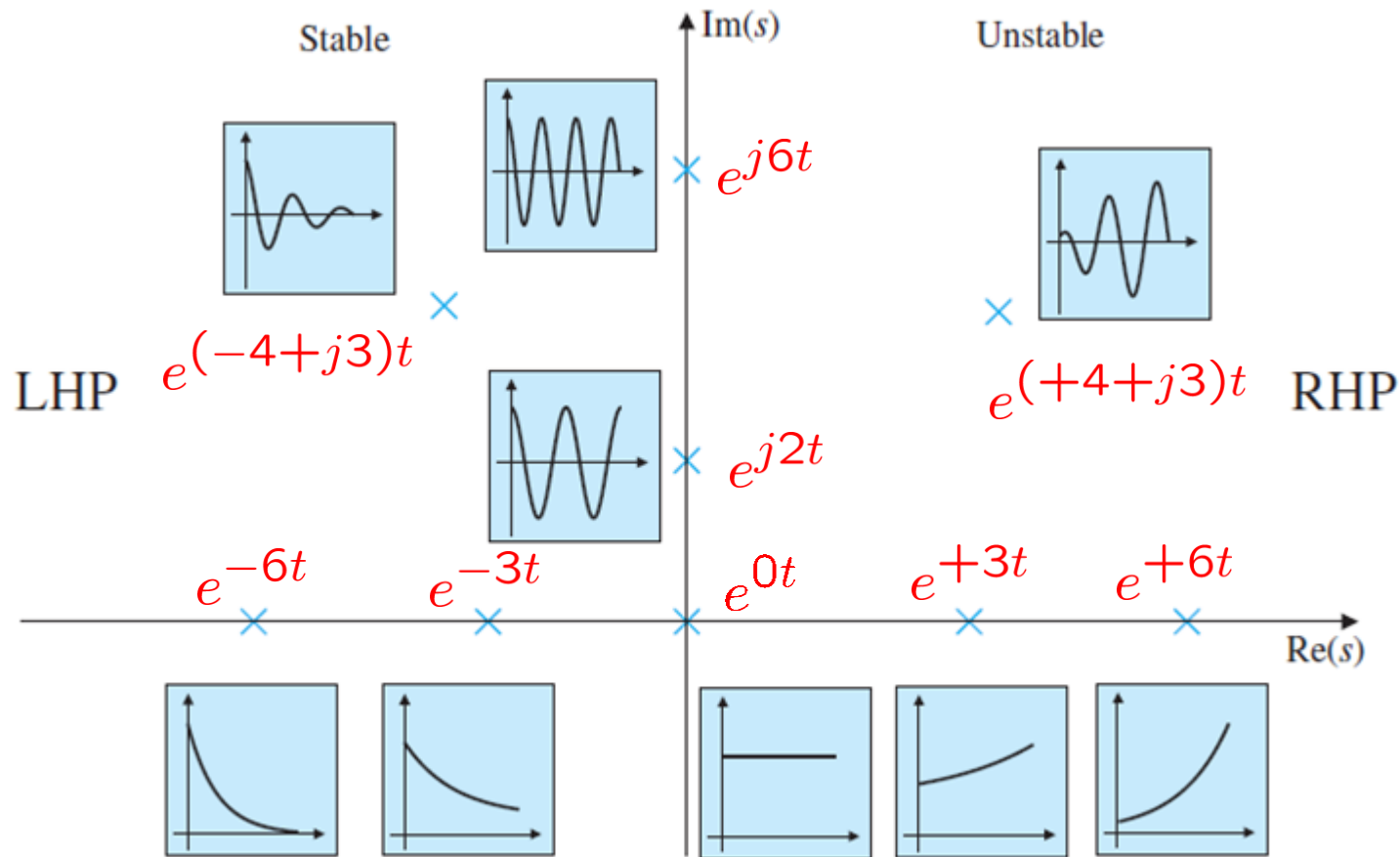
- Resonant Peak:  $M_r$
- Bandwidth:  $\omega_{BW}$

⇒ speed of response

⇒  $\omega_{BW} = \omega_n, \zeta = 0.7$



- Time functions associated with points in the s-plane  
(LHP, left half-plane; RHP, right half-plane)



$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

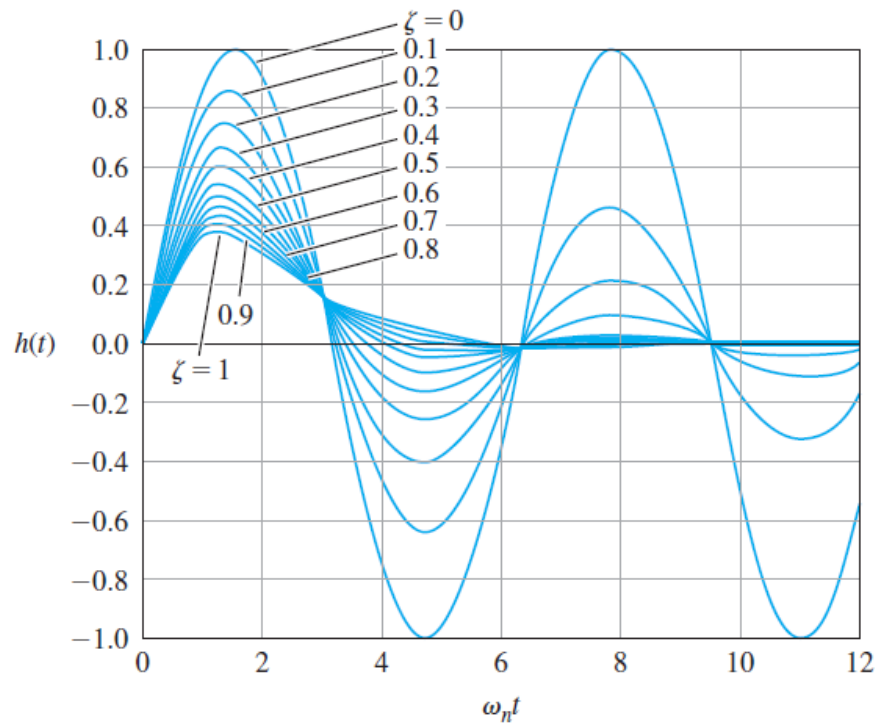
$$\sigma = \omega_n \zeta$$

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} (\sin \omega_d t) 1(t)$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

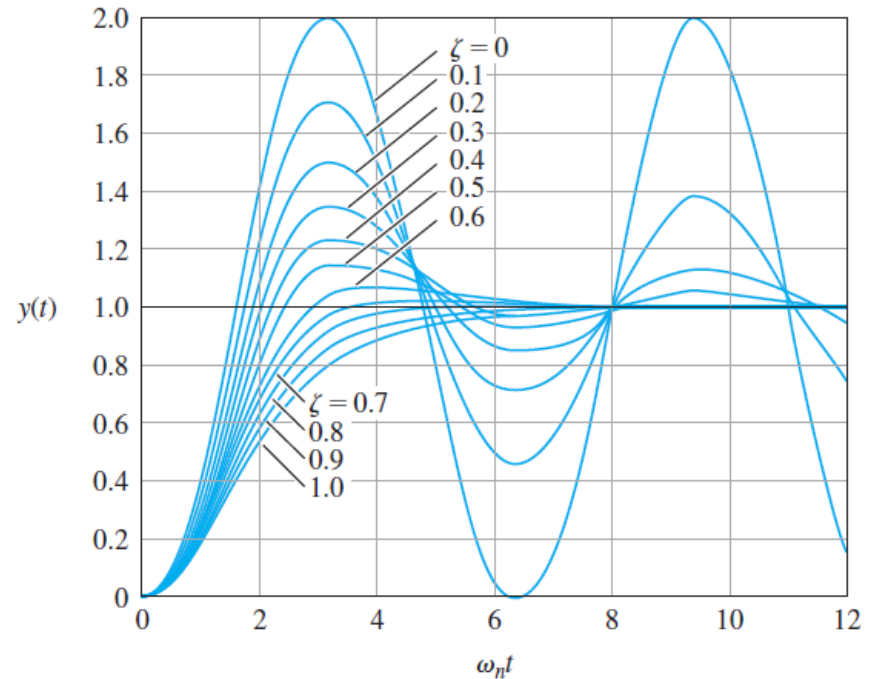
- Responses of second-order systems versus  $\zeta$  :

(a) Impulse Responses



(a)

(b) Step Responses



(b)