

Fall 2021 (110-1)

# 控制系統 Control Systems

## Unit 5C Selected Illustrative Root Loci

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## ■ Example 0: Double Integrator

$$G(s) = \frac{1}{s^2}$$

$$\Rightarrow 1 + K L(s) = 0$$

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0$$

- Satellite attitude, hard-disk drive, motor, etc.

## ● With P controller

$$\Rightarrow 1 + K_P \frac{1}{s^2} = 0$$

## ● Rule 1:

- 2 branches start at  $s = 0$

## ● Rule 2:

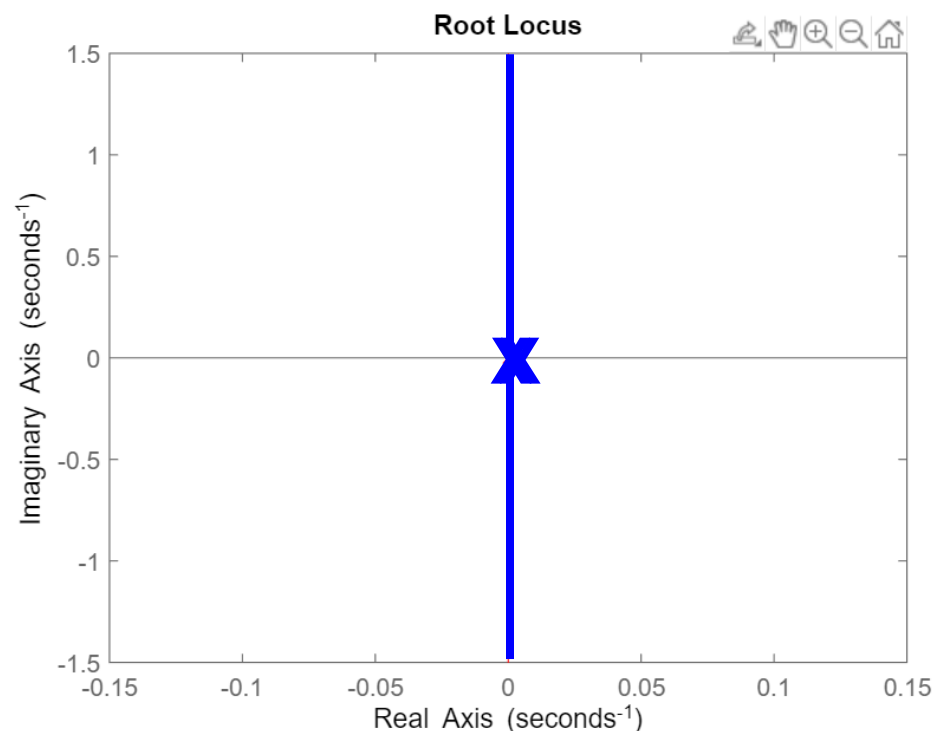
- No locus on real axis

## ● Rule 3:

- Asymptotes:  $\pm 90^\circ$

## ● Rule 4:

- Depart at  $\pm 90^\circ$

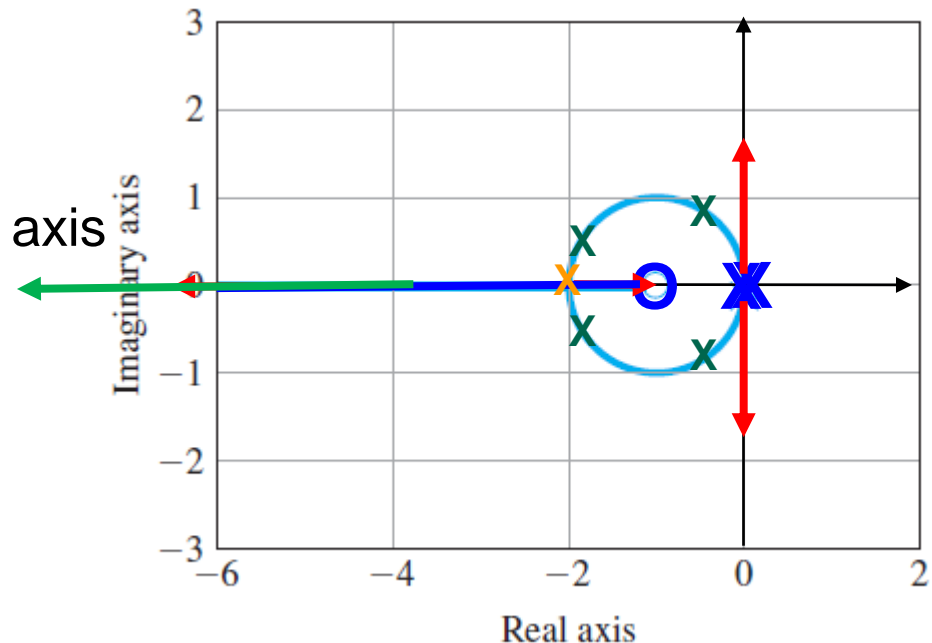


### Example 5.3: Satellite Attitude Control w/ PD Control

$$\Rightarrow 1 + [K_P + K_D s] \frac{1}{s^2} = 0 \quad \Rightarrow 1 + K \frac{b(s)}{a(s)} = 0$$

$$\Rightarrow K = K_D \Rightarrow \frac{K_P}{K_D} = 1 \quad \Rightarrow 1 + K \frac{s+1}{s^2} = 0$$

- Rule 1:
  - Additional zero: pull the locus into the LHP
  - 2 branches start at  $s = 0$ , one to  $s = -1$ , one to  $\infty$
- Rule 2:
  - Real axis: to the left of  $s = -1$
- Rule 3:
  - Asymptotes: along negative real axis
- Rule 4:
  - Depart at  $\pm 90^\circ$
- Rule 5:
  - Rejoin and break at  $s = -2$



## ■ Example 5.3: Satellite Attitude Control w/ PD Control

- Practically, differentiation is not good
- Use the following approximation:

$$\Rightarrow D_c(s) = K_P + \frac{K_D s}{\frac{s}{p} + 1} = K_P + \frac{K_D p s}{s + p}$$

$$= (K_P + pK_D) \frac{s + \frac{pK_P}{K_P + pK_D}}{s + p}$$

$$\Rightarrow 1 + D_c(s) G(s) = 0$$

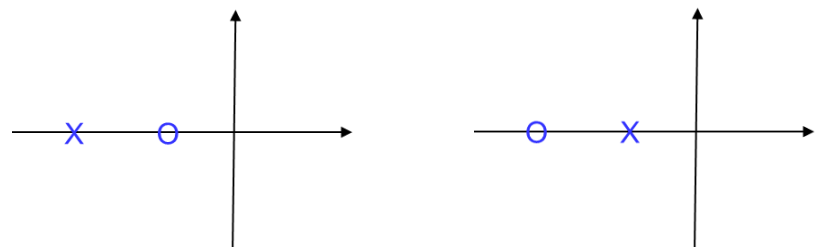
$$\Rightarrow 1 + K L(s) = 0$$

$$= K \frac{s + z}{s + p} \quad \begin{aligned} K &= K_P + pK_D \\ z &= \frac{pK_P}{K_P + pK_D} \end{aligned}$$

$$\Rightarrow 1 + K \frac{(s + z)}{(s + p)} \frac{1}{s^2} = 0$$

● Lead compensator:  $z < p$

● Lag compensator:  $z > p$



## Example 5.4: Satellite Control w/ Modified PD or Lead Compensation

$$z = 1, \quad p = 12 \quad \Rightarrow \quad 1 + K \frac{(s + 1)}{s^2 (s + 12)} = 0$$

### ● Rule 1:

- 3 branches, two start at  $s = 0$ , one at  $s = -12$

### ● Rule 2:

- Real axis:  $-12 \leq s \leq -1$

### ● Rule 3:

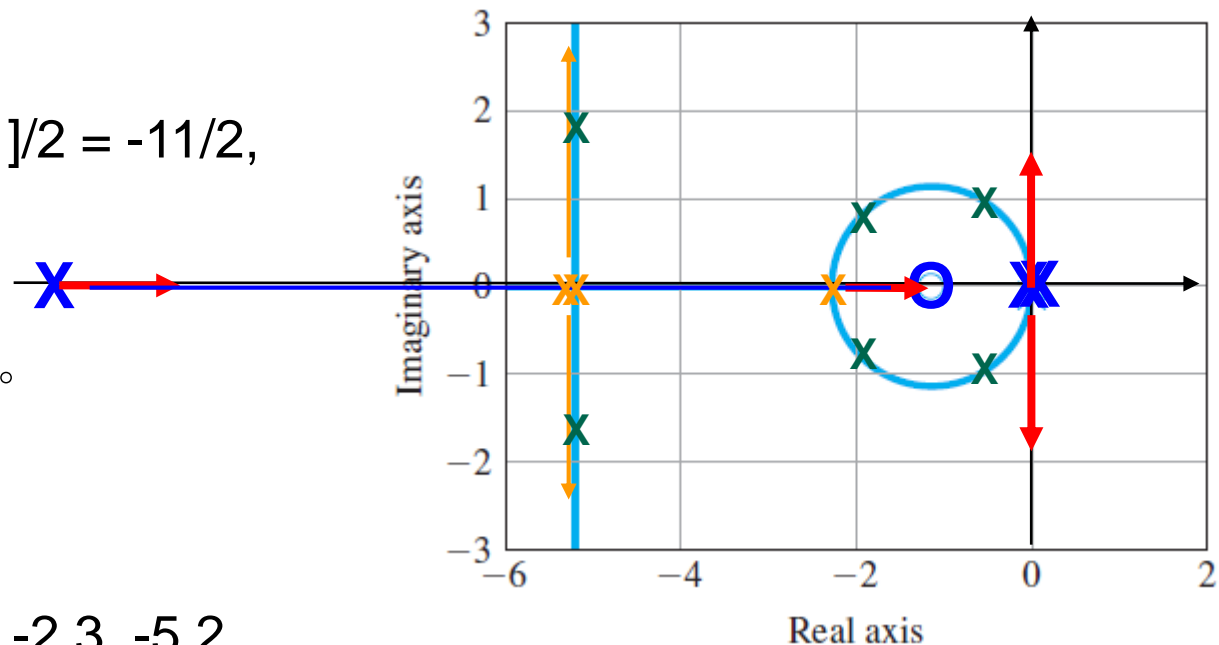
- Asymptotes:  $3 - 1 = 2$ ,  
centered at  $[-12 - (-1)] / 2 = -11/2$ ,  
at  $\pm 90^\circ$

### ● Rule 4:

- Depart at  $s=0$ :  $\pm 90^\circ$
- Depart at  $s=-12$ :  $0^\circ$

### ● Rule 5:

- Multiple roots: at  $s = -2.3, -5.2$



## Example 5.5: Satellite Control w/ Lead Having a Relatively Small Value for the Pole

$$z = 1, \quad p = 4 \quad \Rightarrow \quad 1 + K \frac{(s + 1)}{s^2 (s + 4)} = 0$$

### ● Rule 1:

- 3 branches, two start at  $s = 0$ , one at  $s = -4$

### ● Rule 2:

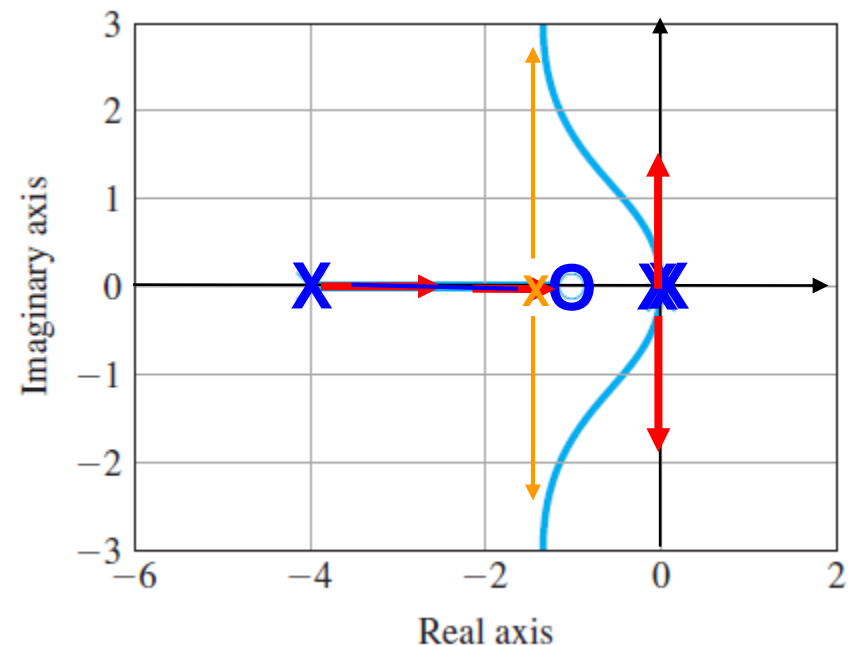
- Real axis:  $-4 \leq s \leq -1$

### ● Rule 3:

- Asymptotes:  $3 - 1 = 2$ ,  
centered at  $[-4 - (-1)] / 2 = -3/2$ ,  
at  $\pm 90^\circ$

### ● Rule 4:

- Depart at  $s=0$ :  $\pm 90^\circ$



## Example 5.6: Satellite w/ Transition Value for the Pole

$$z = 1, \quad p = 9 \quad \Rightarrow \quad 1 + K \frac{(s + 1)}{s^2 (s + 9)} = 0$$

### ● Rule 1:

- 3 branches, two start at  $s = 0$ , one at  $s = -9$

### ● Rule 2:

- Real axis:  $-9 \leq s \leq -1$

### ● Rule 3:

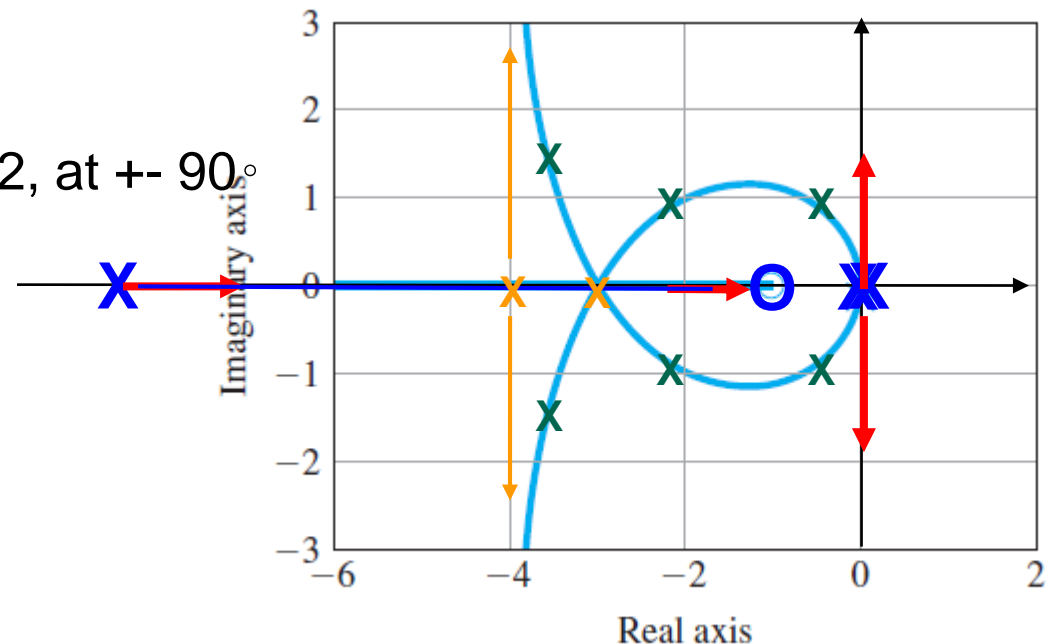
- Asymptotes:  $3 - 1 = 2$ ,  
centered at  $[-9 - (-1)] / 2 = -8/2$ , at  $\pm 90^\circ$ .

### ● Rule 4:

- Depart at  $s=0$ :  $\pm 90^\circ$

### ● At $s = -3$ ,

- arrival:  $+60^\circ, -60^\circ, 0^\circ$
- depart:  $+120^\circ, -120^\circ, 0^\circ$



## Summary of Examples 5.3 5.4, 5.5, 5.6: Satellite Control

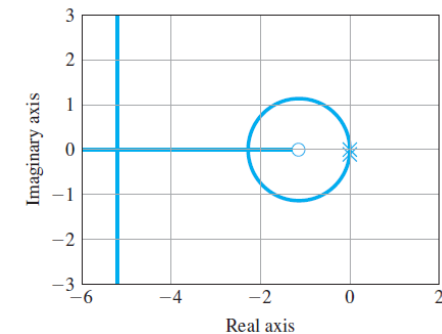
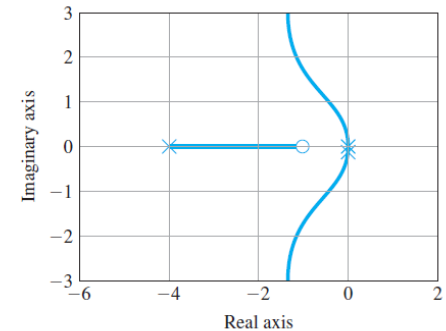
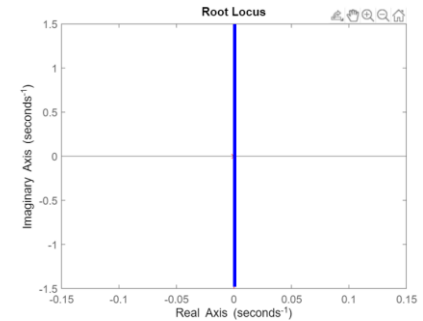
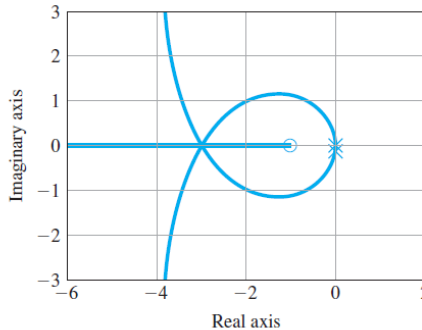
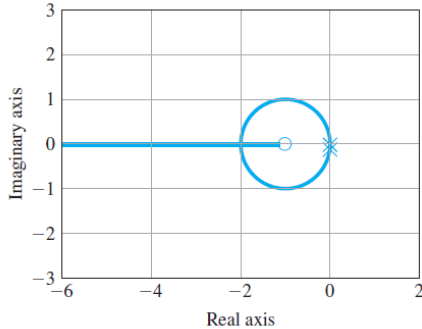
$$\Rightarrow 1 + K_P \frac{1}{s^2} = 0$$

$$\Rightarrow 1 + K \frac{s+1}{s^2} = 0$$

$$\Rightarrow 1 + K \frac{(s+1)}{s^2 (s+4)} = 0$$

$$\Rightarrow 1 + K \frac{(s+1)}{s^2 (s+9)} = 0$$

$$\Rightarrow 1 + K \frac{(s+1)}{s^2 (s+12)} = 0$$



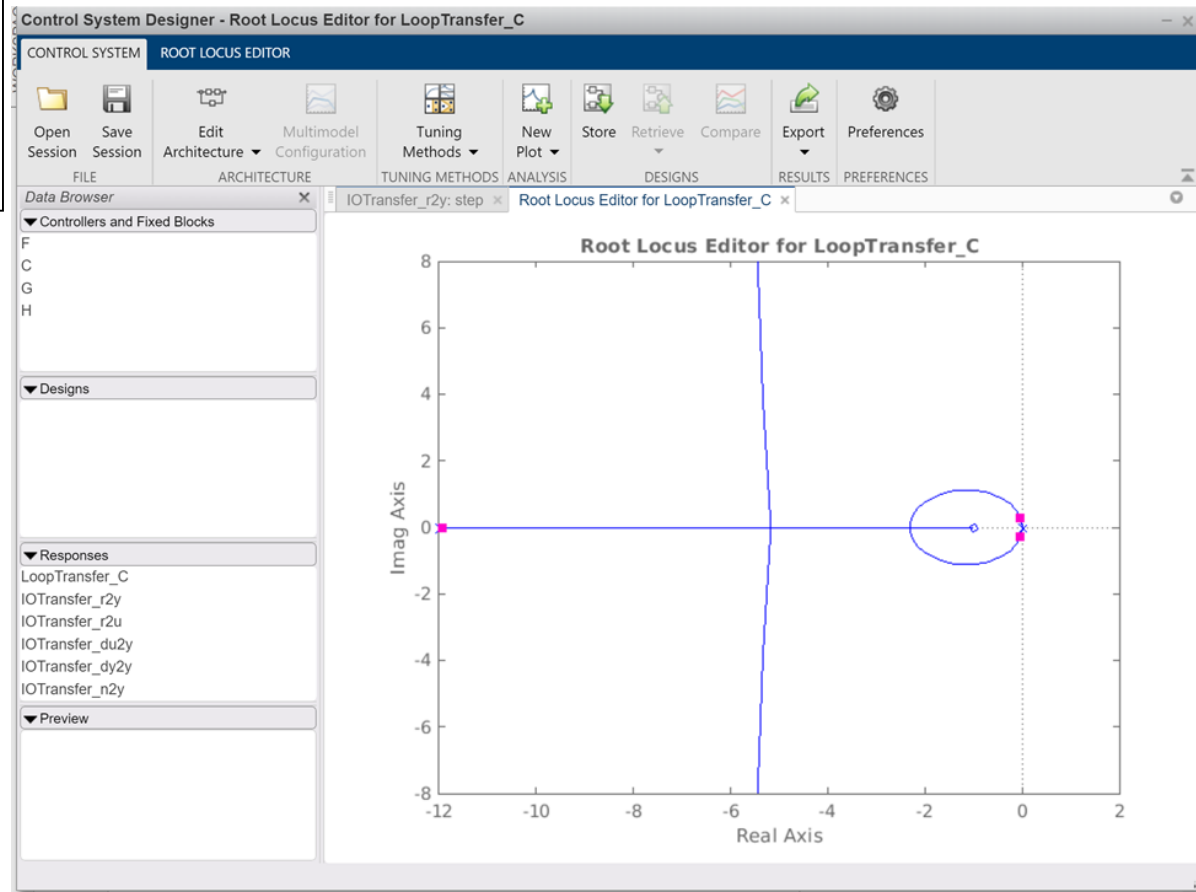


## ■ Example 5.7: Exercise to use rltool (Matlab)

```
s = tf('s')
```

```
sysL = (s+1)/(s^2);  
% sysL = (s+1)/(s^2*(s+12));
```

```
rltool(sysL);  
% sisotool('rlocus', sysL);
```



## Example 5.8: Satellite Control w/ Collocated Flexibility

$$G(s) = \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$

$$\Rightarrow 1 + D_c(s) G(s) = 0 \quad \Rightarrow D_c(s) = K \frac{(s + 1)}{(s + 12)}$$

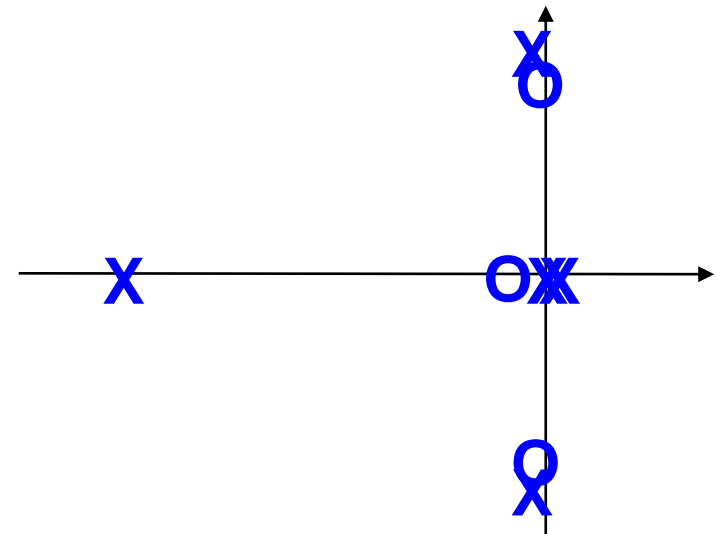
$$\Rightarrow 1 + K L(s) = 0 \quad \Rightarrow L(s) = \frac{(s + 1)}{(s + 12)} \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$

### ● Rule 1:

- 5 branches,
- 3 -> finite zeros
- 2 -> asymptotes

### ● Rule 2:

- Real axis:  $-12 \leq s \leq -1$



## ■ Example 5.8: Satellite Control w/ Collocated Flexibility

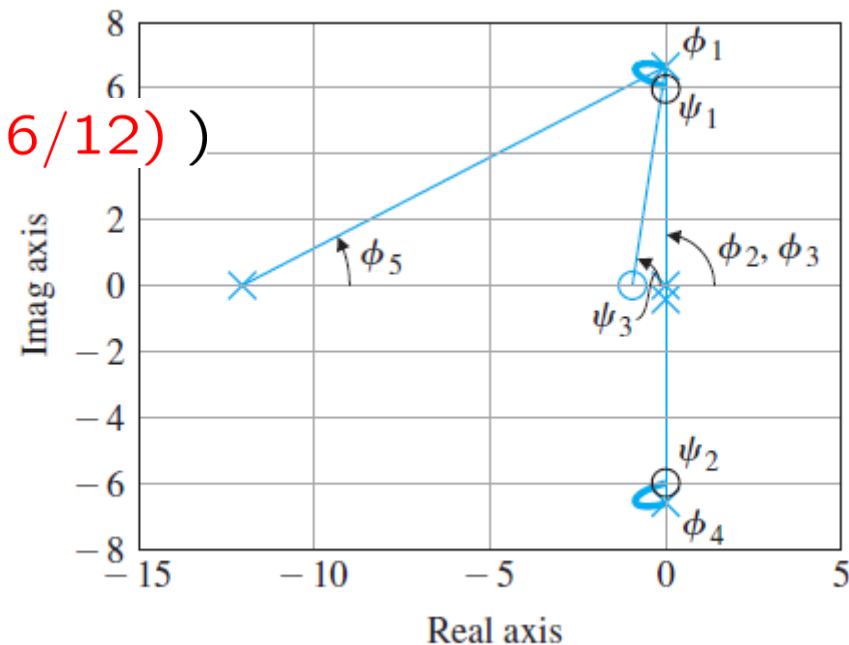
### ● Rule 3:

- Asymptotes:  $5-3 = 2$
- Centered at  $[-12 \ -0.1 \ -0.1 \ -(-0.1 \ -9.1 \ -1)] / 2 = -11/2$ , at  $\pm 90^\circ$

### ● Rule 4:

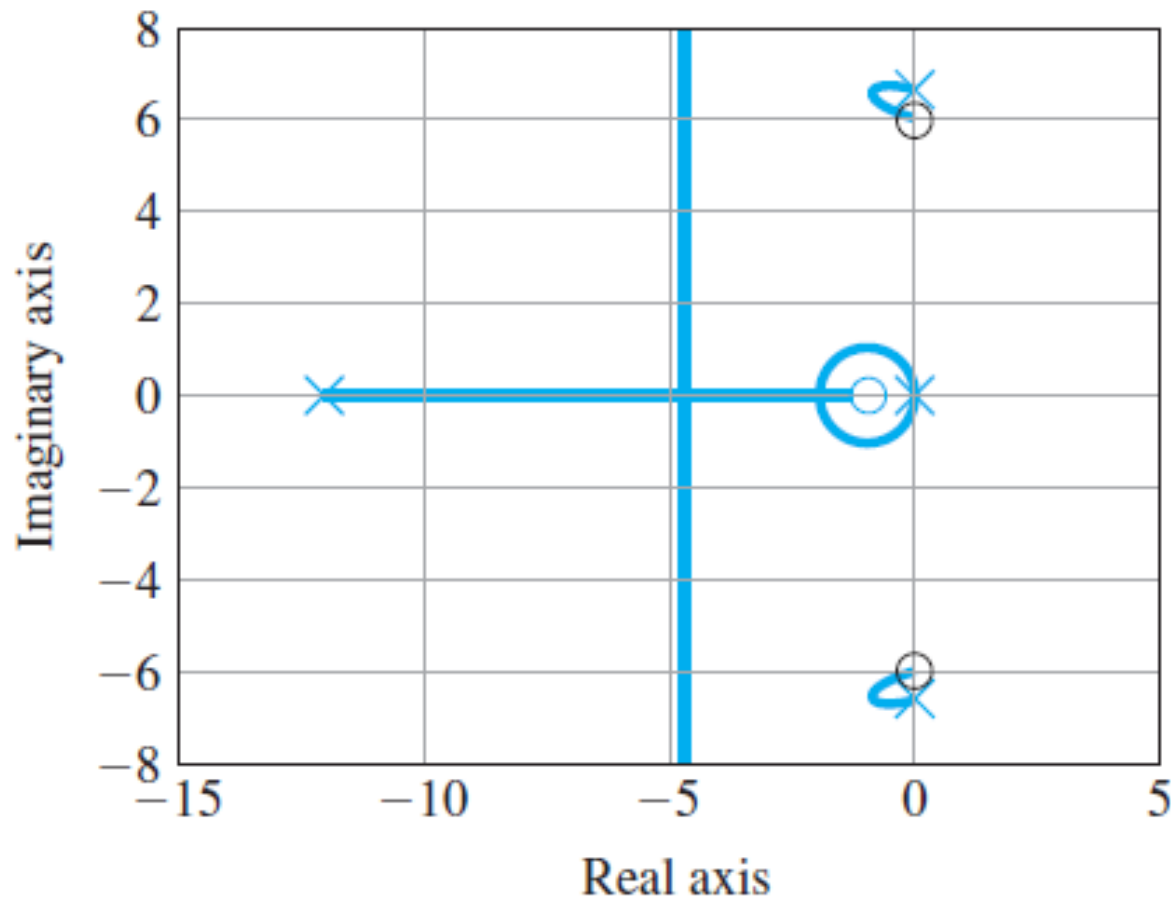
- Depart at  $s = -0.1 + j6.6$ :

$$\begin{aligned}\phi_1 &= \psi_1 + \psi_2 + \psi_3 - (\phi_2 + \phi_3 + \phi_4 + \phi_5) - 180 \\ &= 90 + 90 + \tan^{-1}(6.6) \\ &\quad - (90 + 90 + 90 + \tan^{-1}(6.6/12)) \\ &\quad - 180 \\ &= 142.6\end{aligned}$$



- Example 5.8: Satellite Control w/ Collocated Flexibility

$$\Rightarrow L(s) = \frac{(s + 1)}{(s + 12)} \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$



## Example 5.9: For Non-Collocated Case

$$\Rightarrow K L(s) = D_c(s) G(s) = K \frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$

### ● Rule 1:

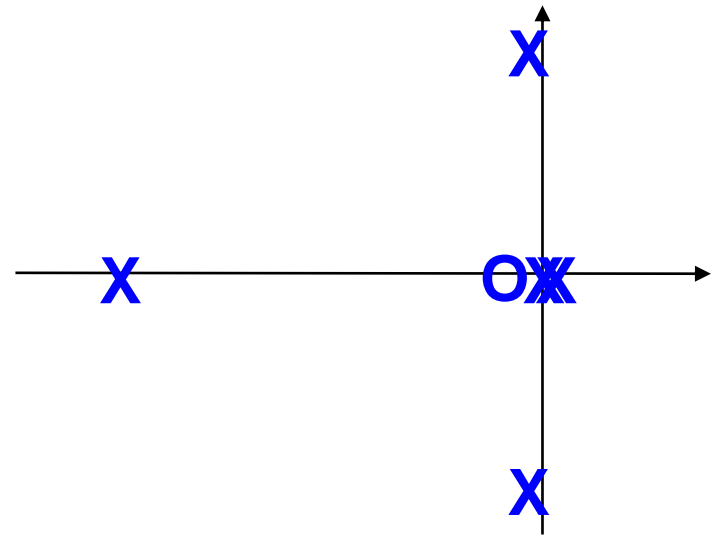
- 5 branches,
- 1 -> finite zero
- 4 -> asymptotes

### ● Rule 2:

- Real axis:  $-12 \leq s \leq -1$

### ● Rule 3:

- Asymptotes:  $5 - 1 = 4$ ,
- Centered at  $[-12 - 0.2 - (-1)] / 4 = -11.2/4$ ,  
at  $\pm 45^\circ$ ,  $\pm 135^\circ$



## Example 5.9: For Non-Collocated Case

$$\Rightarrow K L(s) = D_c(s) G(s) = K \frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$

### ● Rule 4:

- Depart at  $s = -0.1 + j6.6$ :

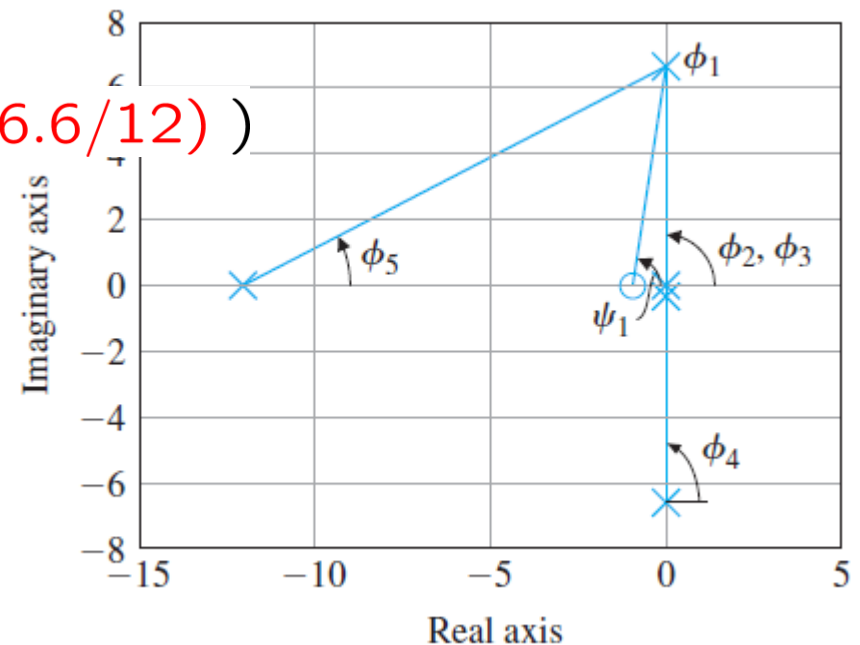
$$\phi_1 = \psi_1 - (\phi_2 + \phi_3 + \phi_4 + \phi_5) - 180$$

$$= \tan^{-1}(6.6)$$

$$- (90 + 90 + 90 + \tan^{-1}(6.6/12))$$

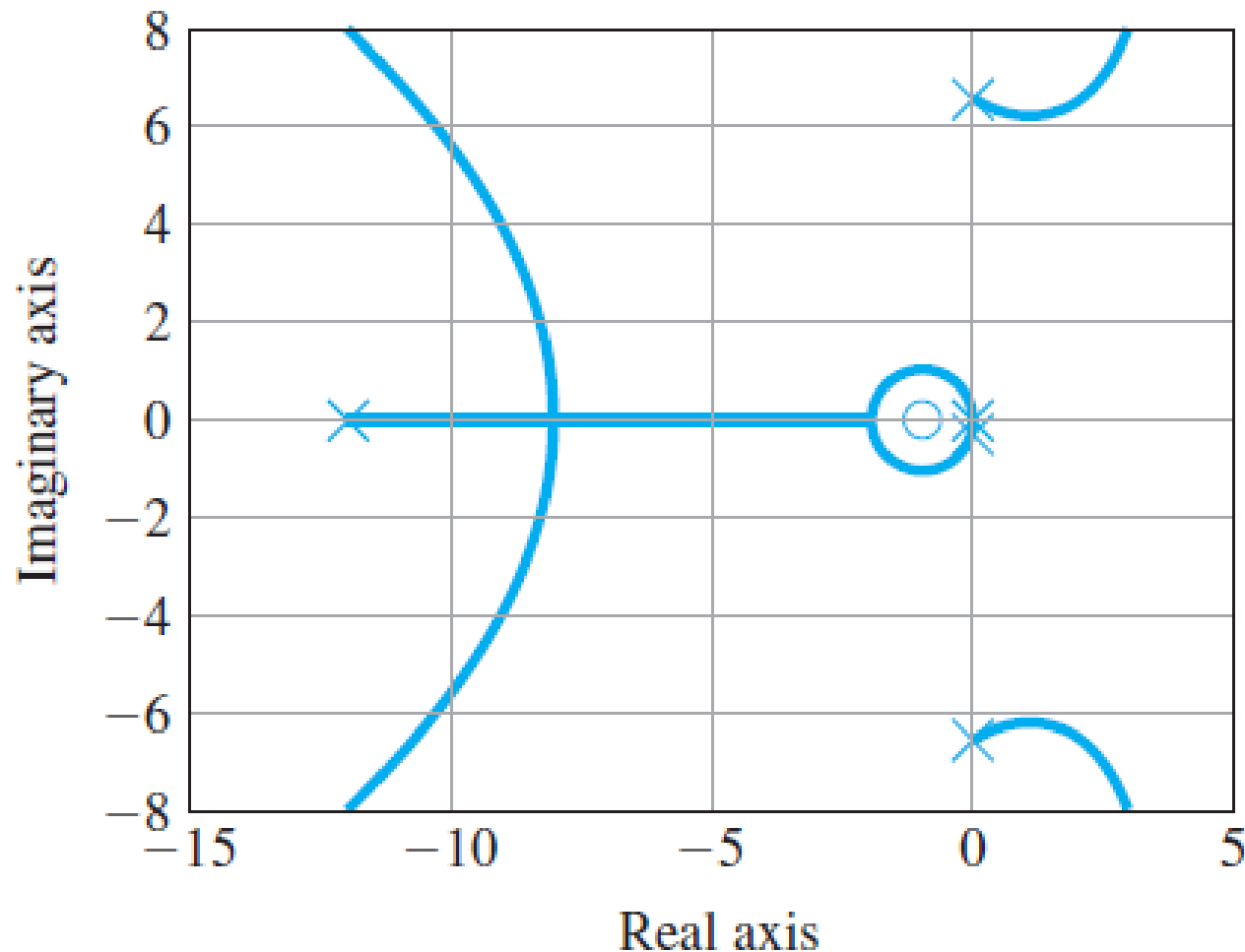
$$- 180$$

$$= -37.4$$



■ Example 5.9: For Non-Collocated Case

$$\Rightarrow K L(s) = D_c(s) G(s) = K \frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$



## Example 5.10: Having Complex Multiple Roots

$$\Rightarrow 1 + K L(s) = 0 \Rightarrow L(s) = \frac{1}{s(s+2)[(s+1)^2 + 4]}$$

### ● Rule 1:

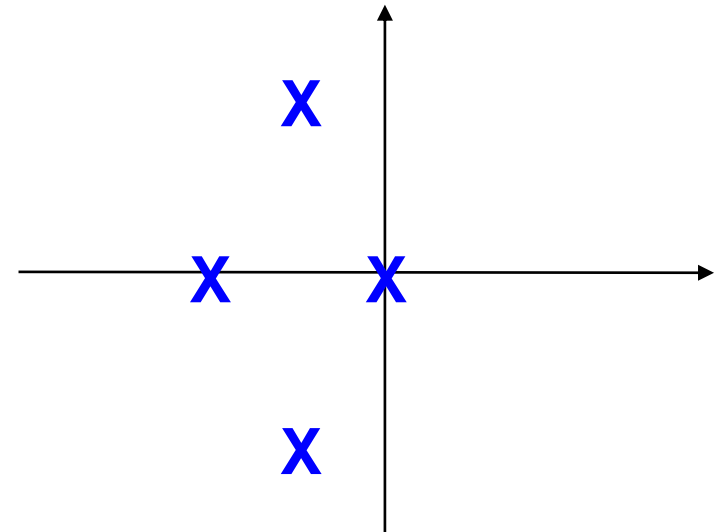
- 4 branches,
- 4  $\rightarrow$  asymptotes

### ● Rule 2:

- Real axis:  $-2 \leq s \leq 0$

### ● Rule 3:

- Asymptotes:  $4 - 0 = 4$ ,
- Centered at  $[-2 - 1 - 1 - 0 + (0)] / 4 = -1$ ,  
at  $\pm 45^\circ, \pm 135^\circ$





## Example 5.10: Having Complex Multiple Roots

$$\Rightarrow 1 + K L(s) = 0 \quad \Rightarrow L(s) = \frac{1}{s(s+2)[(s+1)^2 + 4]}$$

### Rule 4:

- Depart at  $s = -1 + j2$ :

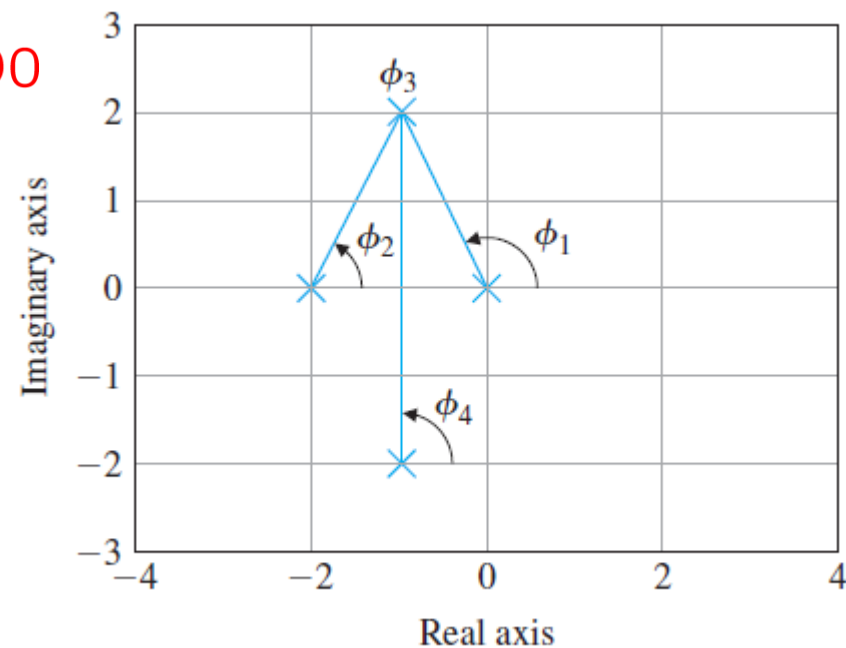
$$\phi_{dep} =$$

$$\phi_3 = -(\phi_1 + \phi_2 + \phi_4) + 180$$

$$= -\tan^{-1}\left(\frac{2}{-1}\right) - \tan^{-1}\left(\frac{2}{1}\right) - 90$$

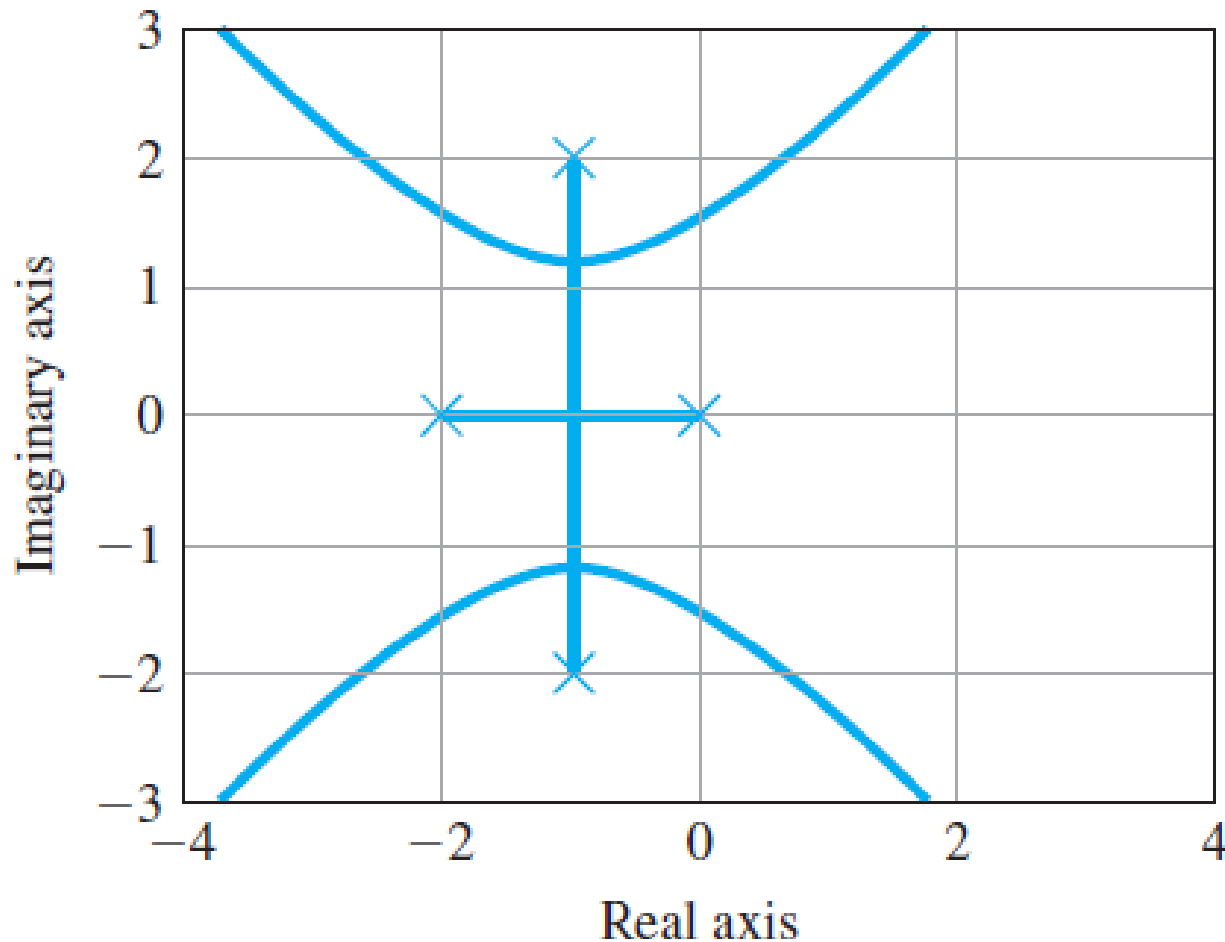
$$+ 180$$

$$= -90$$



- Example 5.10: Having Complex Multiple Roots

$$\Rightarrow 1 + K L(s) = 0 \Rightarrow L(s) = \frac{1}{s(s+2)[(s+1)^2 + 4]}$$



## ■ Examples 5.8, 5.9, 5.10

$$\frac{(s + 1)}{(s + 12)} \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$

$$\frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$

$$\frac{1}{s(s + 2)[(s + 1)^2 + 4]}$$

