

Fall 2021 (110-1)

控制系統
Control Systems

Unit 4B

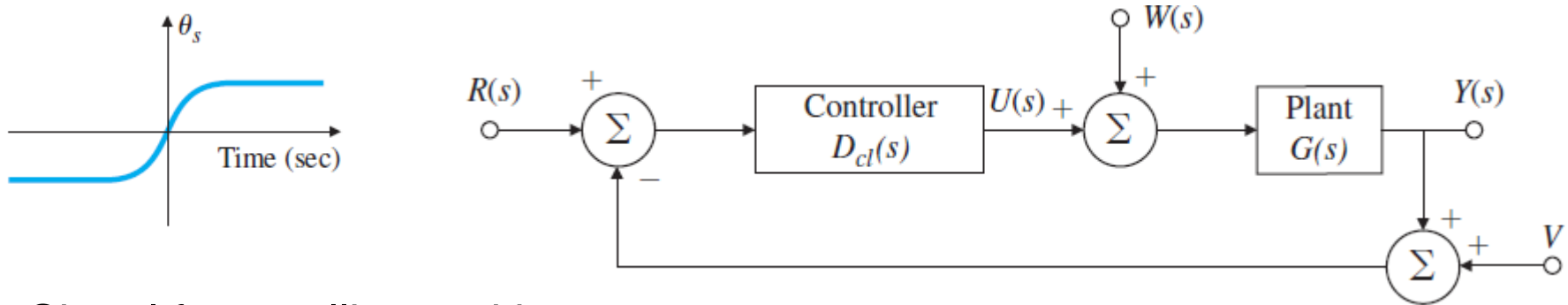
Control of Steady-State Error to Polynomial Inputs:
System Type

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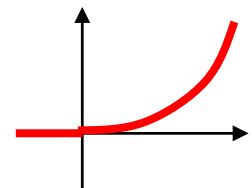
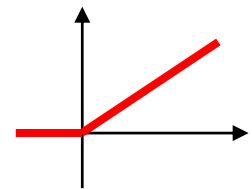
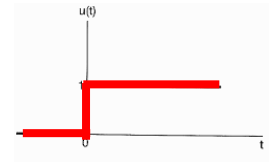
- Closed-loop system:



- Signal for satellite tracking

- Regulation:

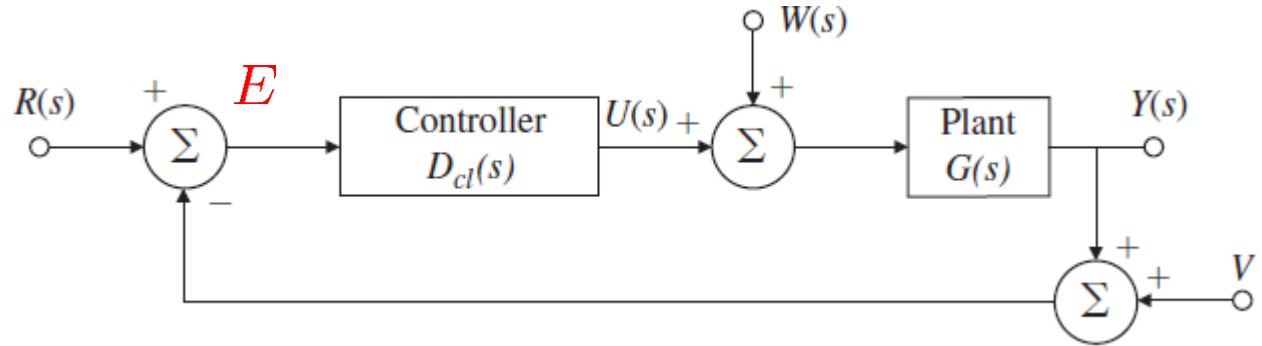
- Reference input is taken to be
 - A constant function,
 - Constant for a long periods of time,
 - A polynomial in time, of low degree.



- System Type:

- The degree of polynomial that they can reasonably track.

● Closed-loop system:



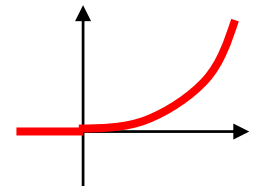
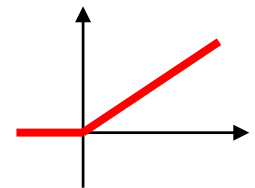
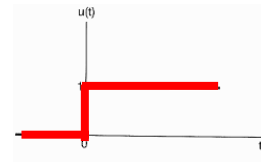
$$E = \frac{1}{1 + G D_{cl}} R$$

$$S = \frac{1}{1 + G D_{cl}}$$

$$= S R$$

$$r(t) = \frac{t^k}{k!} 1(t)$$

$$R(s) = \frac{1}{s^{k+1}}$$



■ Consider polynomial inputs:

■ $k = 0$: step input, position input

■ $k = 1$: ramp input, velocity input

■ $k = 2$: acceleration input

Steady-State Error by the Final-Value Theorem:

$$\begin{aligned}
 e_{ss} &= \lim_{t \rightarrow \infty} e(t) \\
 &= \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + G D_{cl}} R \\
 &= \lim_{s \rightarrow 0} s \frac{1}{1 + G D_{cl}} \frac{1}{s^{k+1}}
 \end{aligned}$$

Type 0:

- If GD has no pole at the origin (no integrator)
- With a unit-step input:

$$\begin{aligned}
 r(t) &= 1(t) \\
 R(s) &= \frac{1}{s}
 \end{aligned}$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + G D_{cl}} \frac{1}{s} = \frac{1}{1 + G D_{cl}(0)}$$

$$\frac{e_{ss}}{r_{ss}} = \frac{e_{ss}}{1} \Rightarrow e_{ss} = \frac{1}{1 + G D_{cl}(0)}$$

Position Error Constant:

$$G D_{cl}(0) \triangleq K_p$$

- In general case:

$$GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n} \quad GD_{cl}^0(s) \text{ has no pole at the origin}$$

$$GD_{cl}^0(0) = K_n$$

- $n = 0$: GD_{cl} has no integrator
- $n = 1$: GD_{cl} has one integrator
- $n = 2$: GD_{cl} has two integrators

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{GD_{cl}^0(s)}{s^n}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} s \frac{1}{\frac{s^n}{s^n} + \frac{GD_{cl}^0(s)}{s^n}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} s \frac{s^n}{s^n + GD_{cl}^0(s)} \frac{1}{s^{k+1}} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

- In general case:
$$e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$
- If $n > k$,
$$e_{ss} = 0$$
- If $n < k$,
$$e_{ss} \rightarrow \infty$$
- If $n = k = 0$,
$$e_{ss} = \frac{1}{1 + K_n}$$
- If $n = k \neq 0$,
$$e_{ss} = \frac{1}{K_n}$$
- Type 0:
 - $n = k = 0$
 - a step (position) input
 - zero-degree polynomial
 - K_0 : position constant, $K_0 = K_p$

- In general case:
$$e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1 + K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

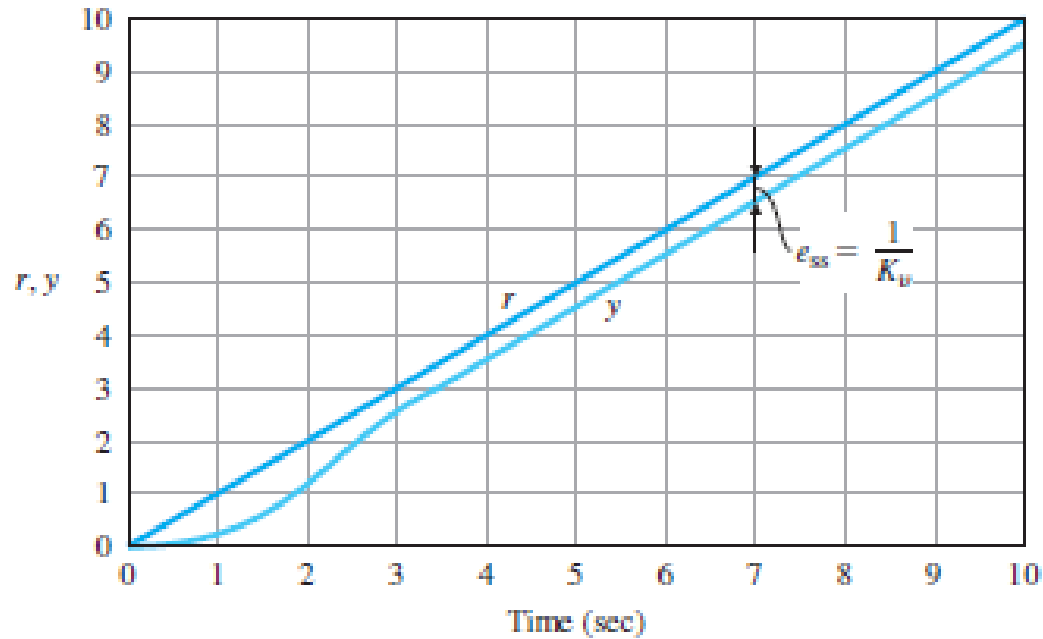
$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

$$GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n}$$

- System Type**: a robust property with respect to parameter changes in the unity feedback structure.

■ In general case: $e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$

● Relationship between ramp response and K_v



Example 4.1: System Type for Speed Control

$$G(s) = \frac{A}{\tau s + 1}$$

$$D(s) = k_P$$

$$\Rightarrow GD_{cl}(s) = \frac{k_P A}{\tau s + 1}$$

\Rightarrow no pole at $s = 0$, $n = 0$

\Rightarrow Type 0, $K_p = k_P A$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + GD_{cl}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{k_P A}{\tau s + 1}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} \frac{\tau s + 1}{\tau s + 1 + k_P A} \frac{1}{s^k}$$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

$$k = 0, \quad e_{ss} = \frac{1}{1 + k_P A}$$

$$= \frac{1}{1 + K_p}$$

$$k = 1, \quad e_{ss} = \infty$$

Example 4.2: System Type using Integral Control

$$G(s) = \frac{A}{\tau s + 1}$$

$$D(s) = k_P + \frac{k_I}{s}$$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

$$\Rightarrow GD_{cl}(s) = \frac{A(k_P s + k_I)}{s(\tau s + 1)}$$

\Rightarrow single pole at $s = 0$, $n = 1$

\Rightarrow Type 1, $K_v = k_I A$

$$k = 0, \quad e_{ss} = 0$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + GD_{cl}} \frac{1}{s^{k+1}}$$

$$k = 1, \quad e_{ss} = \frac{1}{k_I A} = \frac{1}{K_v}$$

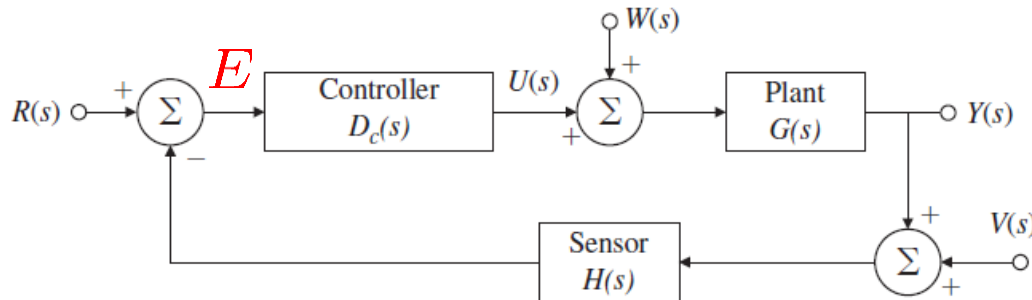
$$= \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{A(k_P s + k_I)}{s(\tau s + 1)}} \frac{1}{s^{k+1}}$$

$$k = 2, \quad e_{ss} = \infty$$

$$= \lim_{s \rightarrow 0} \frac{s(\tau s + 1)}{s(\tau s + 1) + A(k_P s + k_I)} \frac{1}{s^k}$$

- Closed-loop system with sensor dynamics.

R = reference, U = control, Y = output, V = sensor noise



$$\begin{aligned} \frac{Y(s)}{R(s)} &= \mathcal{T}(s) \\ &= \frac{G D_c}{1 + G D_c H} \end{aligned}$$

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s) R(s) = R(s) [1 - \mathcal{T}(s)]$$

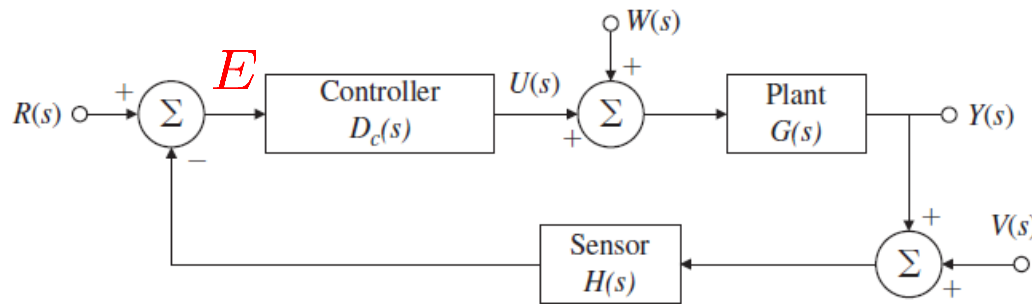
$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s [1 - \mathcal{T}(s)] R(s)$$

- If the reference input a polynomial of degree k:

$$E(s) = \frac{1}{s^{k+1}} [1 - \mathcal{T}(s)]$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1 - \mathcal{T}(s)}{s^{k+1}} = \lim_{s \rightarrow 0} \frac{1 - \mathcal{T}(s)}{s^k}$$

Example 4.3: System Type for Servo w/ Tachometer Feedback



$$G(s) = \frac{1}{s(\tau s + 1)}$$

$$D_c(s) = k_P$$

$$H(s) = 1 + k_t s$$

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s) R(s)$$

$$= R(s) - \frac{G D_c}{1 + H G D_c} R(s) = \frac{1 + (H - 1)G D_c}{1 + H G D_c} R(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} s R(s) [1 - \mathcal{T}(s)] \quad R(s) = \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} \frac{1 - \mathcal{T}(s)}{s^k} = \lim_{s \rightarrow 0} \frac{1}{s^k} \frac{s(\tau s + 1) + (1 + k_t s - 1)k_P}{s(\tau s + 1) + (1 + k_t s)k_P}$$

$$= 0 \quad k = 0$$

$$= \frac{1 + k_t k_P}{k_P}, \quad k = 1$$

- Transfer function from disturbance input $W(s)$ to error $E(s)$

$$\frac{E(s)}{W(s)} = -\frac{Y(s)}{W(s)} = \mathcal{T}_w(s) \quad \text{with } R(s) = 0$$

$$\mathcal{T}_w(s) = s^n \mathcal{T}_{w,0}(s) \quad \text{with } \mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$$

$$\begin{aligned} y_{ss} &= \lim_{s \rightarrow 0} s \mathcal{T}_w(s) \frac{1}{s^{k+1}} \\ &= \lim_{s \rightarrow 0} \mathcal{T}_{w,0}(s) \frac{s^n}{s^k} \end{aligned}$$

- If $n > k$,

$$y_{ss} = 0$$

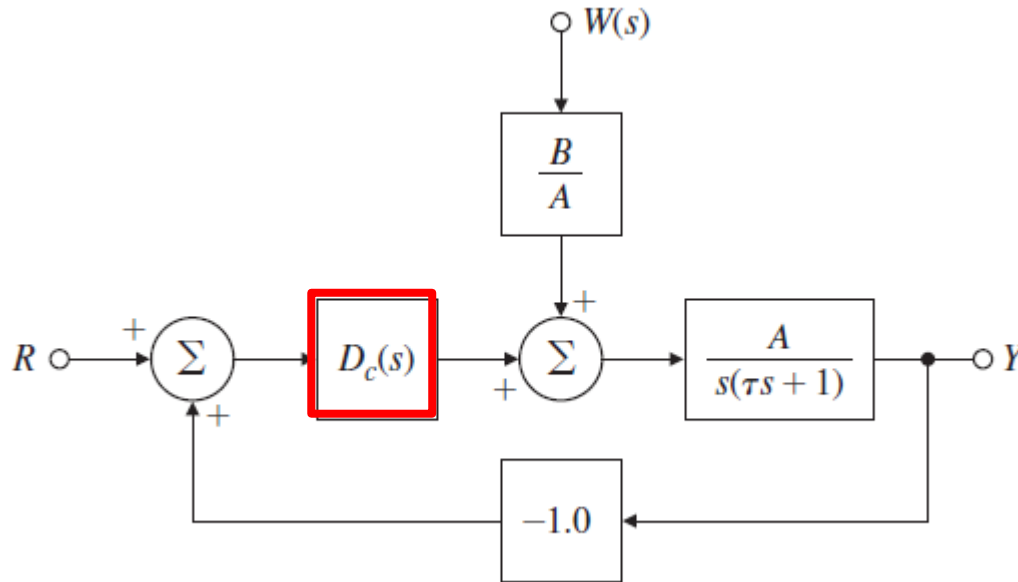
- If $n < k$,

$$y_{ss} \rightarrow \infty$$

- If $n = k$,

$$y_{ss} = \frac{1}{K_{w,n}}$$

- Example 4.4: System Type for DC Motor Position Control



(a) $D_c(s) = k_P$

(b) $D_c(s) = k_P + \frac{k_I}{s}$

Example 4.4: System Type for DC Motor Position Control

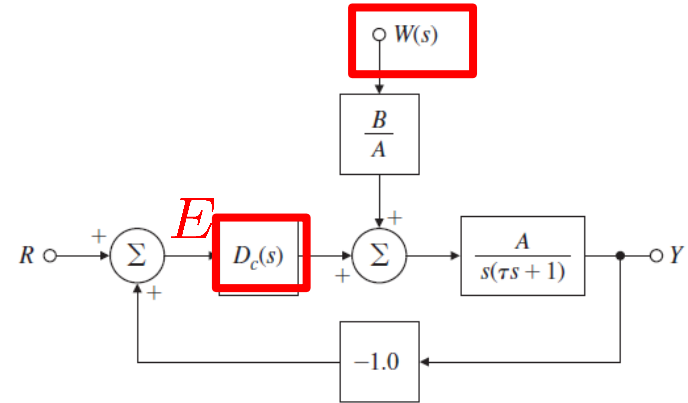
(a) $D_c(s) = k_P$

Transfer function from W to E ($R=0$)

$$\mathcal{T}_w(s) = -\frac{B}{s(\tau s + 1) + A k_P}$$

$$= s^0 \mathcal{T}_{w,0}(s) \quad n = 0 \quad \mathcal{T}_w(s) = s^n \mathcal{T}_{w,0}(s)$$

$$K_{w,0} = -\frac{A k_P}{B} \quad \text{with } \mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$$



Type 0: Steady-state error to a unit-step torque input is:

$$e_{ss} = -\frac{B}{A k_P}$$

$$y_{ss} = \frac{1}{K_{w,n}}$$

Steady-state error to a unit-ramp torque input is: $e_{ss} = -\infty$

Example 4.4: System Type for DC Motor Position Control

(b) $D_c(s) = k_P + \frac{k_I}{s}$

Transfer function from W to E ($R=0$)

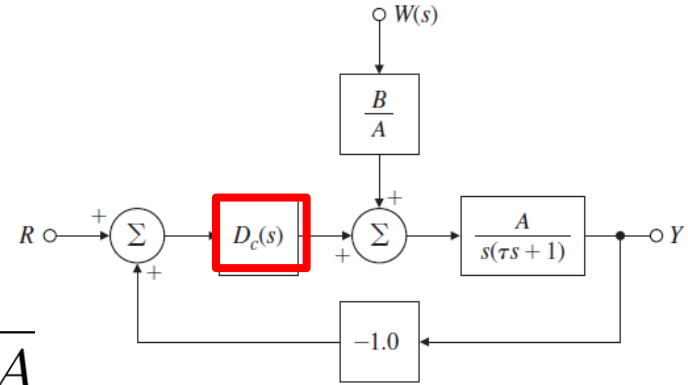
$$T_w(s) = - \frac{Bs}{s^2(\tau s + 1) + (k_P s + k_I)A}$$

$$n = 1$$

$$K_{w,n} = - \frac{A k_I}{B}$$

$$T_w(s) = s^n T_{w,0}(s)$$

$$\text{with } T_{w,0}(0) = \frac{1}{K_{w,n}}$$



Type 1: Steady-state error to a **unit-ramp** disturbance input is:

$$e_{ss} = - \frac{B}{A k_I}$$

$$y_{ss} = \frac{1}{K_{w,n}}$$

Steady-state error to a **unit t^2** disturbance input is: $e_{ss} = -\infty$

Steady-state error to a **unit-step** disturbance input is: $e_{ss} = 0$

■ In general case:
$$e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1 + K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0 \qquad GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n}$$

$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

- **System Type**: a robust property with respect to parameter changes in the unity feedback structure.