Fall 2021 (110-1)

控制系統
Control Systems

Unit 3F
Stability

Feng-Li Lian

NTU-EE

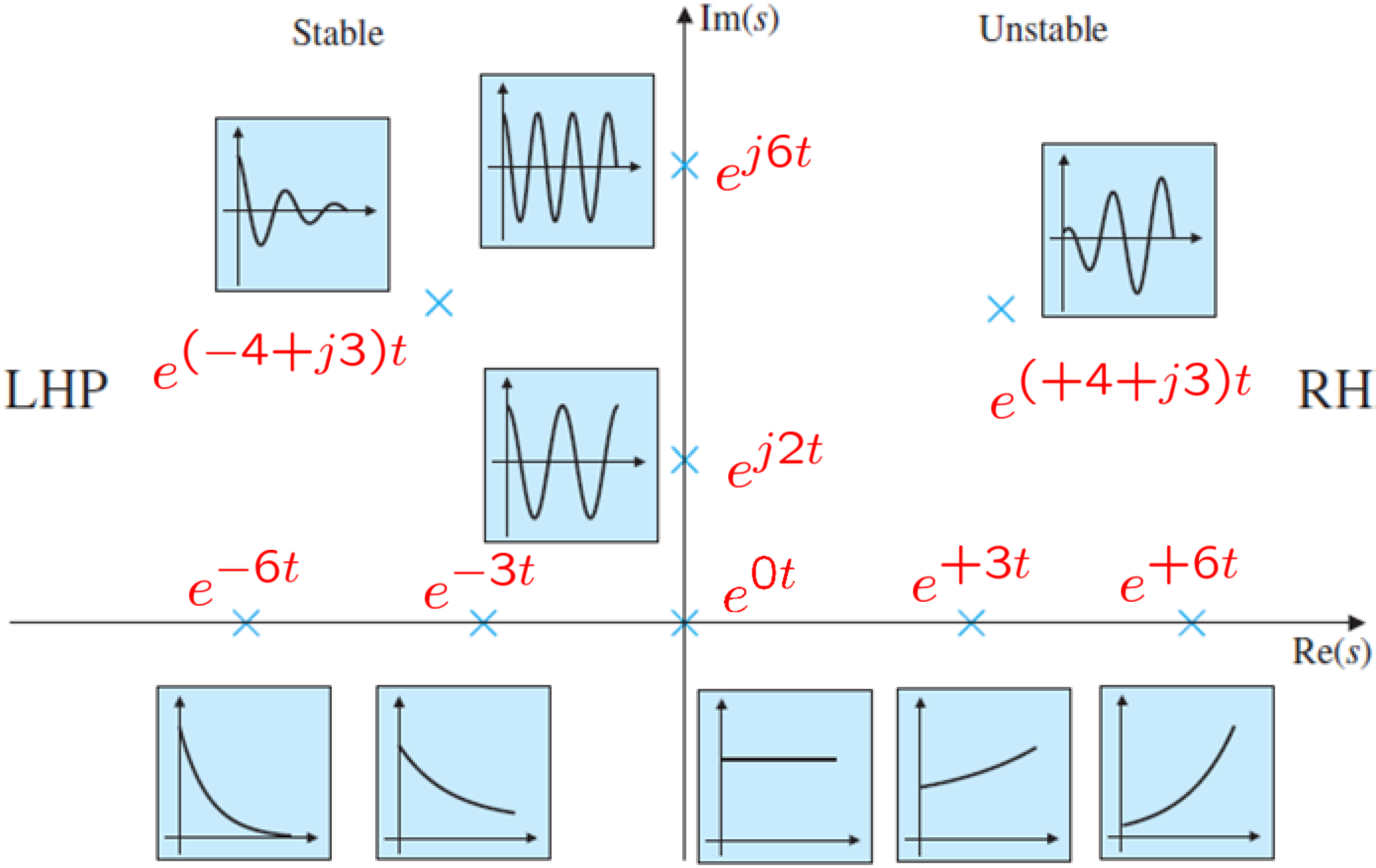
Sep 2021 – Jan 2022

- Stability:
  - An LTI system is said to be stable
  - if all the roots of the transfer function denominator polynomial
  - have negative real parts

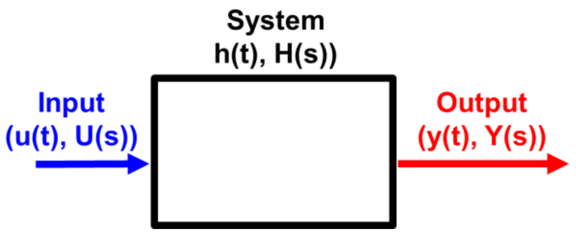
    (that is, they are all in the left-hand s-plane)
  - and is unstable otherwise.

- Stable System:
  - A system is stable
  - if its initial conditions decay to zero
  - and is unstable if they diverge.

- Time functions associated with points in the s-plane
  (LHP, left half-plane; RHP, right half-plane)



- Bounded Input-Bounded Output Stability (BIBO Stable)
  - A system is said to have BIBO stability
  - if every bounded input results in a bounded output (regardless of what goes on inside the system).

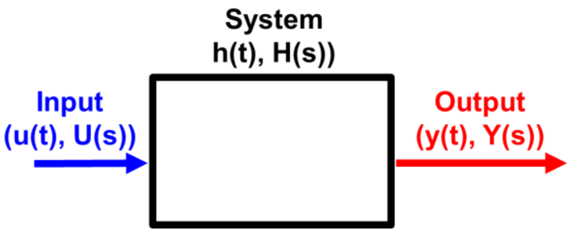


- If the system has input u(t), output y(t), and impulse response h(t), then

$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$

Bounded Input-Bounded Output Stability

$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$



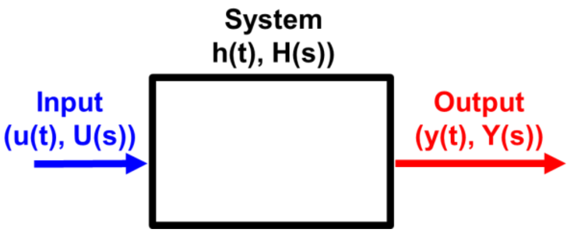
- If input u(t) is bounded $\quad |u(\cdot)| \leq M < \infty$
- And output y(t) is bounded by

$$|y| = \left| \int_{-\infty}^{\infty} h\, u\, d\tau \right| \leq \int_{-\infty}^{\infty} |h|\, |u|\, d\tau \leq M \int_{-\infty}^{\infty} |h|\, d\tau$$

- That is, output y(t) is bounded if $\displaystyle\int_{-\infty}^{\infty} |h|\, d\tau$ is bounded.

Bounded Input-Bounded Output Stability

$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$



- On the other hand,

  suppose the integral is not bounded and the input is bounded,
- e.g., $u(t-\tau) = +1$ if $h(\tau) > 0$

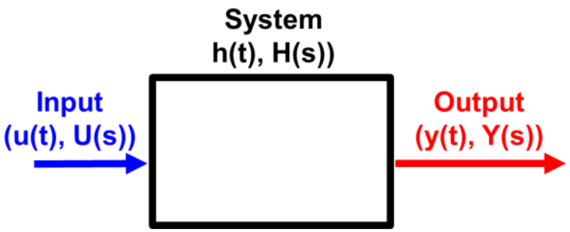
  and $u(t-\tau) = -1$ if $h(\tau) < 0$

$$|y| = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

- The output y(t) is not bounded.

Bounded Input-Bounded Output Stability

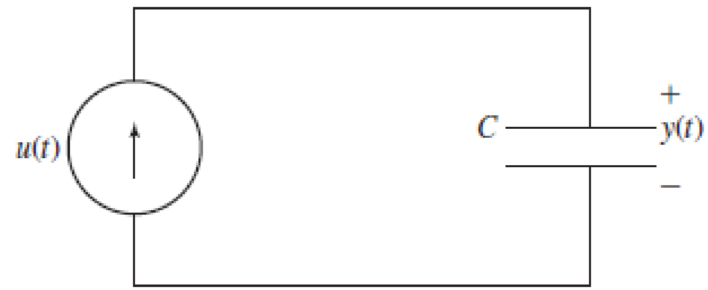
$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$



- Mathematical Definition of BIBO Stability
  - The system with impulse response h(t) is BIBO stable
  - if and only if

    the integral
    $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Example 3.31: BIBO Stability for a Capacitor



- Capacitor driven by current source

$$\Rightarrow h(\tau) = 1(t)$$

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau = \int_{-\infty}^{\infty} d\tau \rightarrow \infty$$

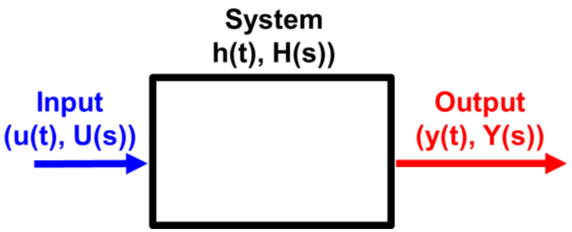
- Consider the LTI whose transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}$$

$$= K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}, \quad m \le n$$

$$\Rightarrow \; y(t) = \sum_{i=1}^{n} K_i \, e^{p_i t}$$

- p_i are the roots of a(s), denominator polynomial
- K_i depend on the initial condition and zero locations



- The system is stable
- if and only if (necessary and sufficient condition)
- every term goes to zero as time goes to infinity

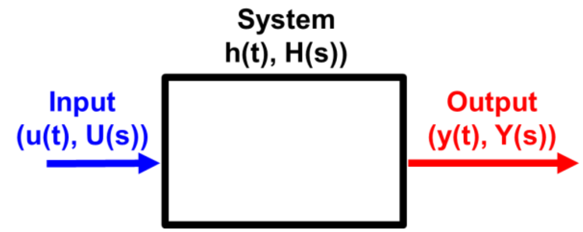
$$\Rightarrow \quad e^{p_i t} \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad \text{for all } p_i$$

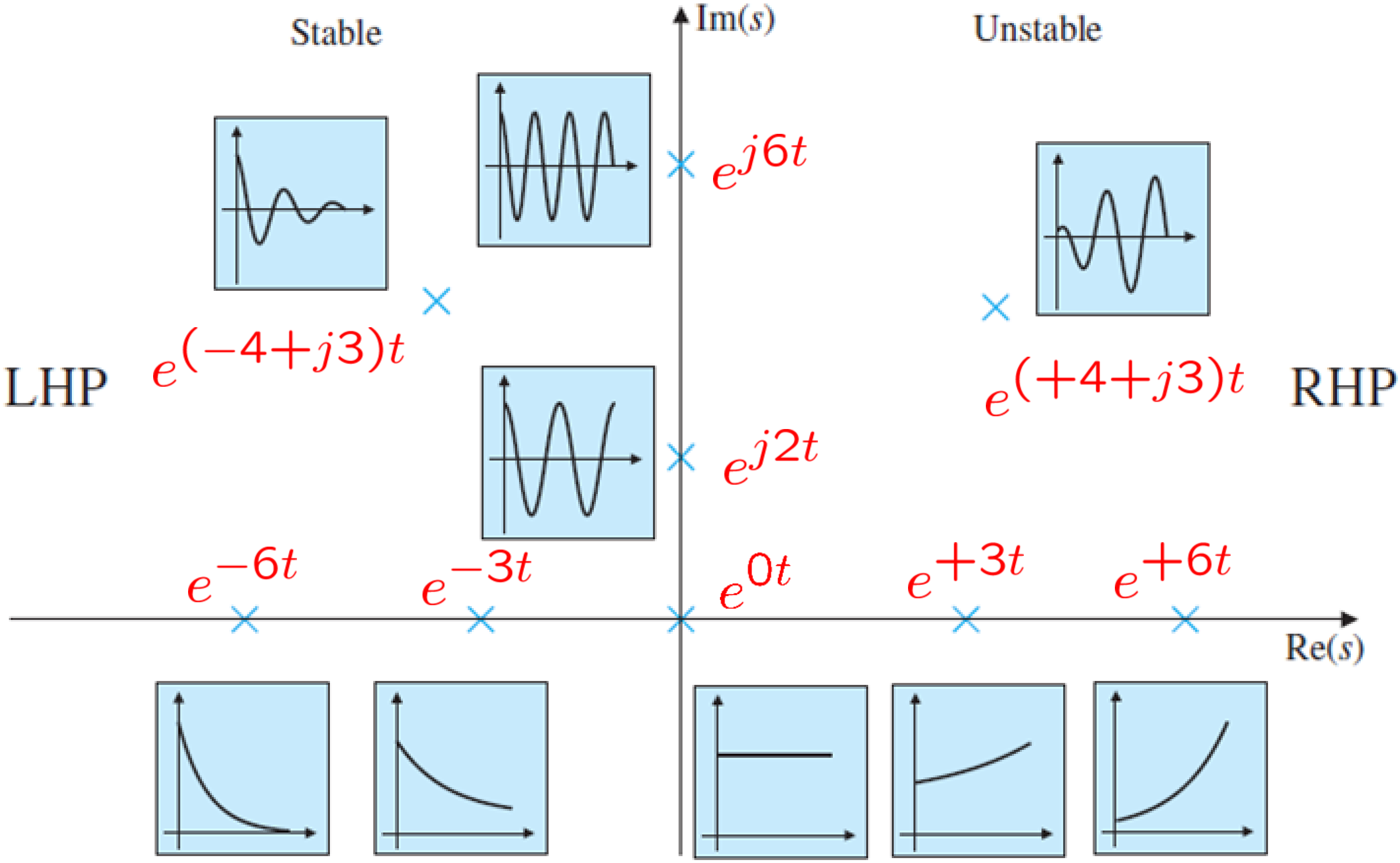
- This will happen
- if all the poles of the system are strictly in the LHP:

$$\Rightarrow \quad \text{Re}\{\, p_i \,\} < 0$$

- This is called internal stability

Stability of LTI Systems





stable

neutrally stable

unstable

- Consider the characteristic equation of an nth-order system:

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- A necessary (but not sufficient) condition

  for stability of the system is that

- all of the roots have negative real parts

- which in turn requires that

  all the $\{ a_i \}$ be positive.

- Equivalent test were independently proposed

  by Routh in 1874 and Hurwitz in 1895.

- Routh showed that
- a system is stable if and only if
- all the elements in the first column of the Routh array
- are positive.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$$

- Routh array:

$$\begin{array}{lll}
\text{Row } n & s^n & : \; 1 \quad a_2 \quad a_4 \quad \cdots \\
\text{Row } n-1 & s^{n-1} & : \; a_1 \quad a_3 \quad a_5 \quad \cdots \\
\text{Row } n-2 & s^{n-2} & : \; b_1 \quad b_2 \quad b_3 \quad \cdots \\
\text{Row } n-3 & s^{n-3} & : \; c_1 \quad c_2 \quad c_3 \quad \cdots
\end{array}$$

$$b_1 = -\frac{\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1}$$

$$b_2 = -\frac{\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$b_3 = -\frac{\det \begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1}$$

- Routh showed that
- a system is stable if and only if
- all the elements in the first column of the Routh array
- are positive.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- Routh array:

$$\begin{array}{lll}
\text{Row } n & s^n & : \; 1 \quad a_2 \quad a_4 \quad \cdots \\
\text{Row } n-1 & s^{n-1} & : \; a_1 \quad a_3 \quad a_5 \quad \cdots \\
\text{Row } n-2 & s^{n-2} & : \; b_1 \quad b_2 \quad b_3 \quad \cdots \\
\text{Row } n-3 & s^{n-3} & : \; c_1 \quad c_2 \quad c_3 \quad \cdots
\end{array}$$

$$c_1 = -\frac{\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1}$$

$$c_2 = -\frac{\det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

$$c_3 = -\frac{\det \begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1}$$

Routh's Stability Criterion

- Routh showed that
- a system is stable if and only if
- all the elements in the first column of the Routh array
- are positive.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- Routh array:

$$\begin{array}{llclllll}
\text{Row } n & s^n & : & 1 & a_2 & a_4 & \cdots \\
\text{Row } n-1 & s^{n-1} & : & a_1 & a_3 & a_5 & \cdots \\
\text{Row } n-2 & s^{n-2} & : & b_1 & b_2 & b_3 & \cdots \\
\text{Row } n-3 & s^{n-3} & : & c_1 & c_2 & c_3 & \cdots \\
\text{Row } \vdots & \vdots & : & & & & \\
\text{Row } 2 & s^2 & : & * & * & & \\
\text{Row } 1 & s & : & * & & & \\
\text{Row } 0 & s^0 & : & * & & &
\end{array}$$

▪ Example 3.32: Routh's Test

$$a(s) = \boxed{s^6} + 4s^5 + \boxed{3s^4} + 2s^3 + \boxed{s^2} + 4s + \boxed{4}$$

$$
\begin{array}{lcllll}
s^6 & : & 1 & 3 & 1 & 4 \\
s^5 & : & 4 & 2 & 4 & 0
\end{array}
$$

$$s^4 \; : \; -\frac{1\cdot 2 - 4\cdot 3}{4} = \boxed{\frac{5}{2}} \qquad -\frac{4\cdot 1 - 4\cdot 1}{4} = 0 \qquad -\frac{1\cdot 0 - 4\cdot 4}{4} = 4$$

$$s^3 \; : \; -\frac{4\cdot 0 - 5/2\cdot 2}{5/2} = \boxed{2} \qquad -\frac{4\cdot 4 - 5/2\cdot 4}{5/2} = -\frac{12}{5}$$

$$s^2 \; : \; -\frac{5/2\cdot(-12/5) - 2\cdot 0}{2} = \boxed{3} \qquad -\frac{5/2\cdot 0 - 2\cdot 4}{2} = 4$$

$$s \; : \; -\frac{5/2\cdot(-12/5) - 2\cdot 0}{2} = \boxed{-\frac{76}{15}} \qquad 0$$

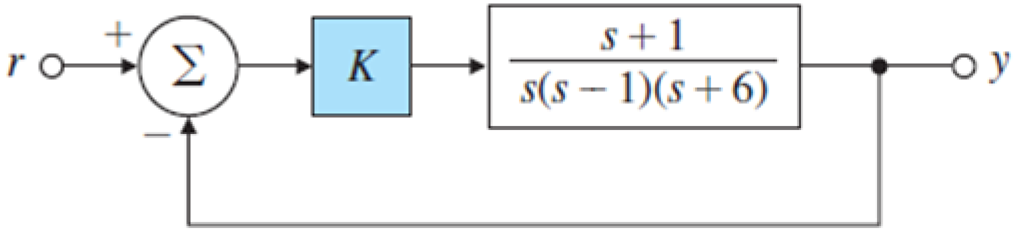
$$s^0 \; : \; -\frac{0 - (-76/15)\cdot 4}{(-76/15)} = \boxed{4}$$

```
roots( [ 1 4 3 2 1 4 4 ] )

ans =
-3.2644 + 0.0000i
 0.6797 + 0.7488i
 0.6797 - 0.7488i
-0.6046 + 0.9935i
-0.6046 - 0.9935i
-0.8858 + 0.0000i
```

Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability



The characteristic equation for the system:

$$1 + K\frac{s+1}{s(s-1)(s+6)} = 0 \qquad \boxed{s^3} + 5s^2 + \boxed{(K-6)s} + K = 0$$

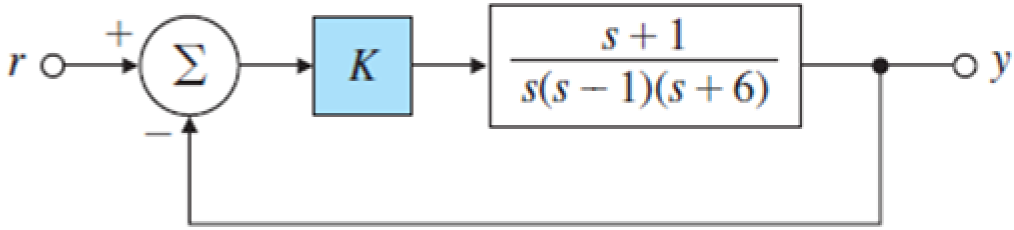
$$\begin{array}{lcll} s^3 & : & 1 & K-6 \\ s^2 & : & 5 & K \\ s & : & (4K-30)/5 & \\ s^0 & : & K & \end{array}$$

$$\Rightarrow \frac{(4K-30)}{5} > 0 \qquad \Rightarrow K > 7.5$$

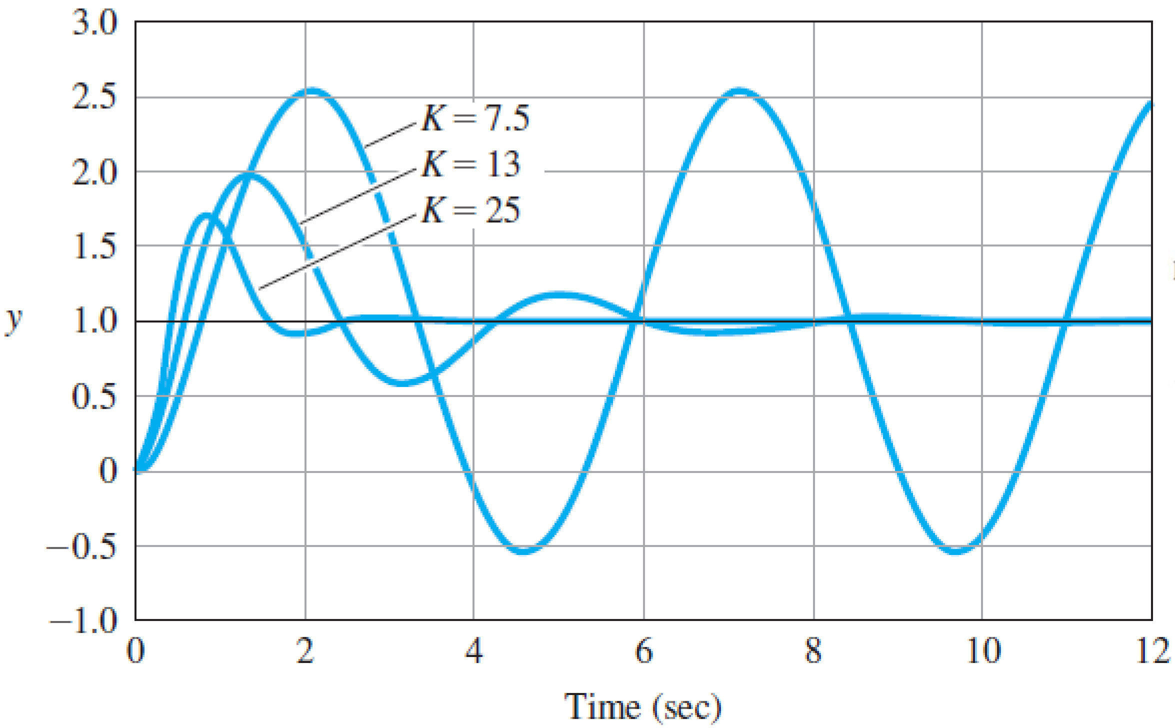
$$\Rightarrow K > 0$$

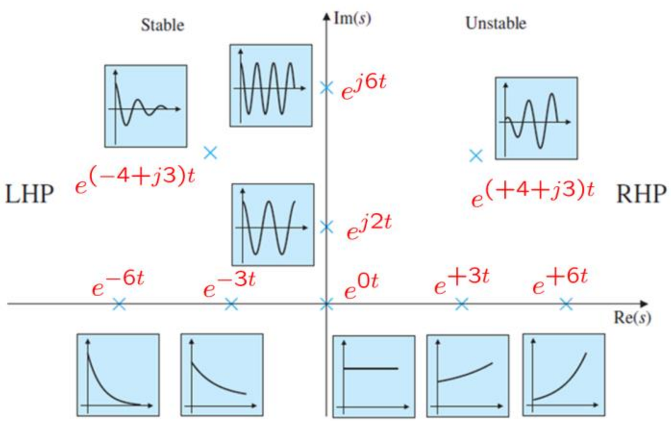
Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability




```
set_K = [ 25  13  7.5  5 ];
for i = 1:4
k = set_K(i);
sysCL = k*(s+1)/(s^3+5*s^2+(k-6)*s+k)
y = step( sysCL, t);
end
```

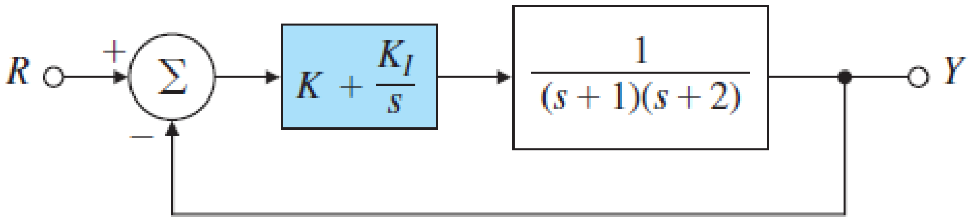




- Transient responses for the system

- Example 3.34: Stability versus Two Parameter Ranges

- System with proportional-integral (PI) control



- The characteristic equation for the system:

$$1 + \left(K + \frac{K_I}{s}\right)\frac{1}{(s+1)(s+2)} = 0$$

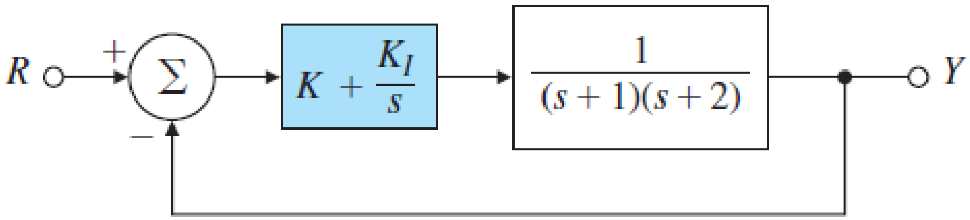
$$\boxed{s^3} + 3s^2 + \boxed{(2+K)s} + K_I = 0$$

$$\begin{array}{lcll} s^3 & : & 1 & 2+K \\ s^2 & : & 3 & K_I \end{array}$$

$$\begin{array}{lcl} s & : & (6+3K-K_I)/3 \\ s^0 & : & K_I \end{array} \quad \begin{array}{l} \Rightarrow \; K > \frac{1}{3}K_I - 2 \\ \Rightarrow \; K_I > 0 \end{array}$$

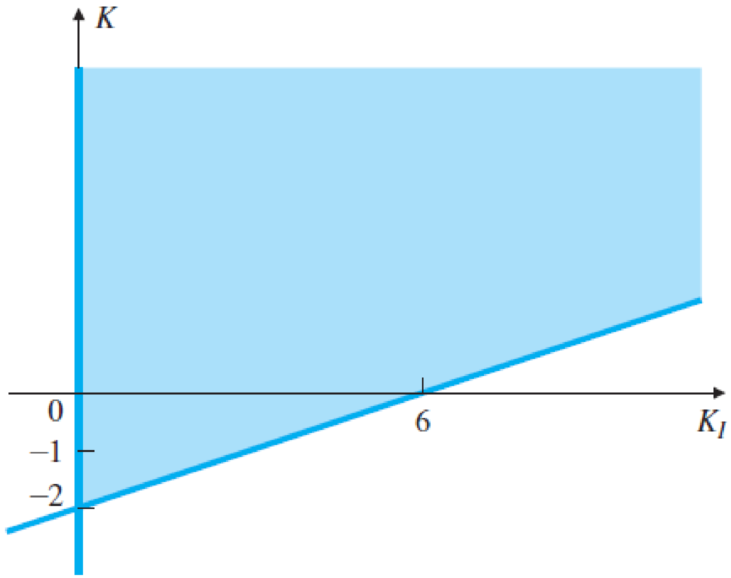
Example 3.34: Stability versus Two Parameter Ranges

- System with proportional-integral (PI) control



$$\Rightarrow \quad K > \frac{1}{3} K_I - 2$$

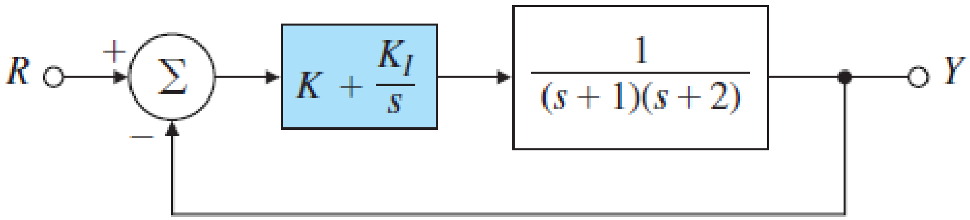
$$\Rightarrow \quad K_I > 0$$



- Allowable region for stability

Example 3.34: Stability versus Two Parameter Ranges

- System with proportional-integral (PI) control



$$\Rightarrow \quad K > \frac{1}{3} K_I - 2$$

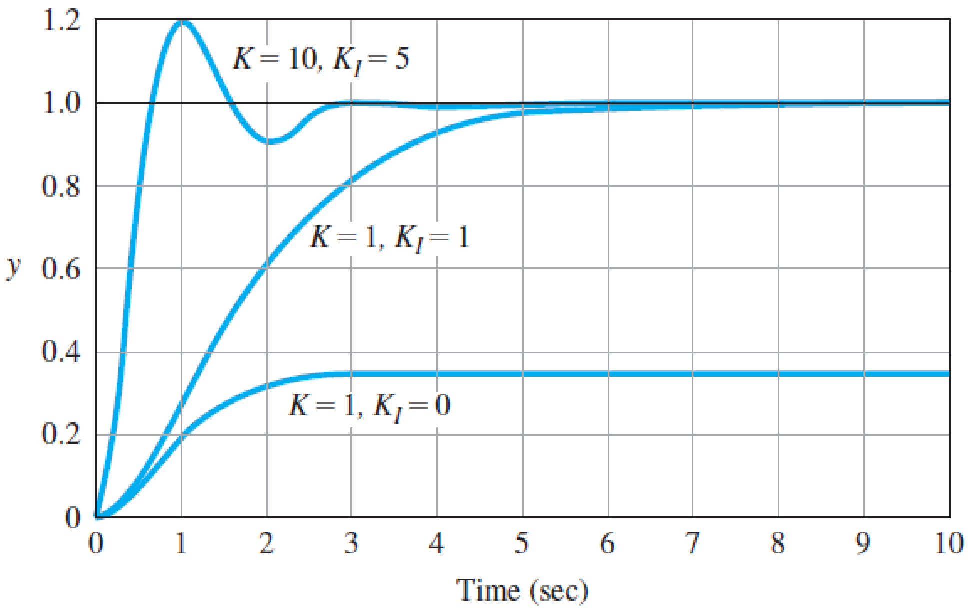
$$\Rightarrow \quad K_I > 0$$

```
set_KI = [ 0 1 5];
set_K = [ 1 10 ];

for i = 1:3
for j = 1:2
KI = set_KI(i);
K = set_K(j);

sysCL = (K*s+KI)/(s^3+3*s^2+(2+K)*s+KI);
y = step( sysCL, t );

end
end
```



- Transient response for the system