Fall 2021 (110-1)

控制系統
Control Systems

Unit 3C
Effect of Pole Locations

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NTU-EE

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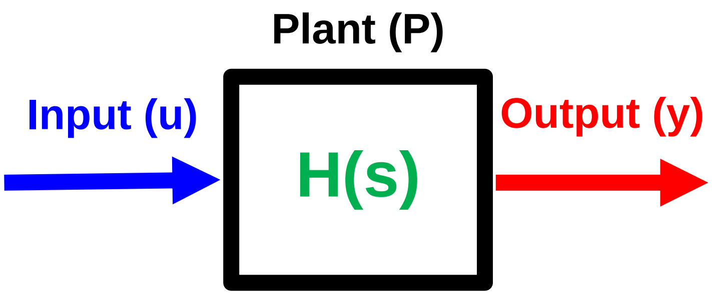
Effect of Pole Locations

- Transfer Function = a ration of polynomials:

$$H(s) = \frac{b(s)}{a(s)}$$

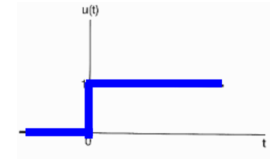


- a(s) and b(s) have **NO** common factors
- Poles: roots of a(s) = 0
- Zeros: roots of b(s) = 0

Effect of Pole Locations

- Transfer Function and Impulse Response:

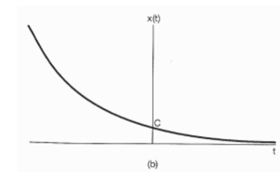
$$H(s) = \frac{1}{s+a} \quad \Rightarrow \quad h(t) = e^{-at}\, 1(t)$$



- a>0: pole: s = -a, at negative s,

  the exponential expression decays

  the impulse response is stable

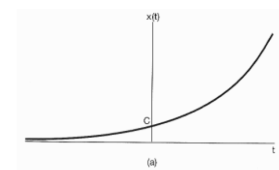


$1(t),\ t \geq 0$

- a<0: pole: s = -a, at positive s,

  the exponential expression grows

  the impulse response is unstable



$1(t),\ t \geq 0$

- The impulse response is the natural response of the system

Effect of Pole Locations

- Transfer Function, Impulse Response, Time Constant

$$H(s) = \frac{1}{s+a} \qquad \Rightarrow \tau \triangleq \frac{1}{a}$$

$$\Rightarrow h(t) = e^{-at}\, 1(t) \qquad \text{at } t = \tau \qquad \Rightarrow h(\tau) = \frac{1}{e}$$

- Impulse Response:

$$u(t) = \delta(t) \qquad \Rightarrow U(s) = 1$$

$$Y(s) = H(s)\,U(s) = \frac{1}{s+a}\,1 = \frac{1}{s+a}$$

$$\Rightarrow y(t) = e^{-at}\, 1(t)$$

- Step Response:

$$u(t) = 1(t) \qquad \Rightarrow U(s) = \frac{1}{s}$$

$$Y(s) = H(s)\,U(s) = \frac{1}{s+a}\,\frac{1}{s} = \frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

$$\Rightarrow y(t) = \frac{1}{a}(1 - e^{-at})\, 1(t)$$

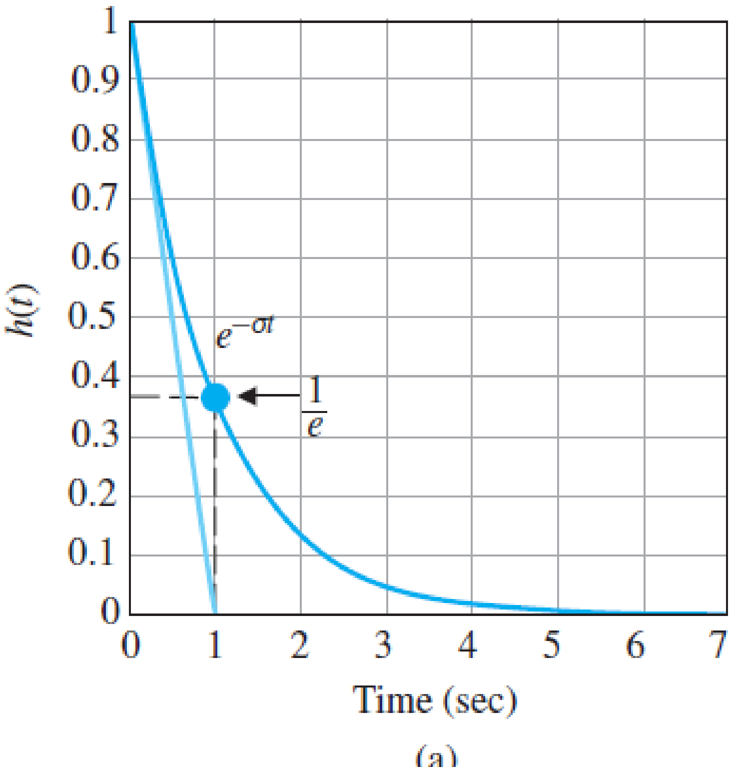
Effect of Pole Locations

- Transfer Function, Impulse Response, Time Constant

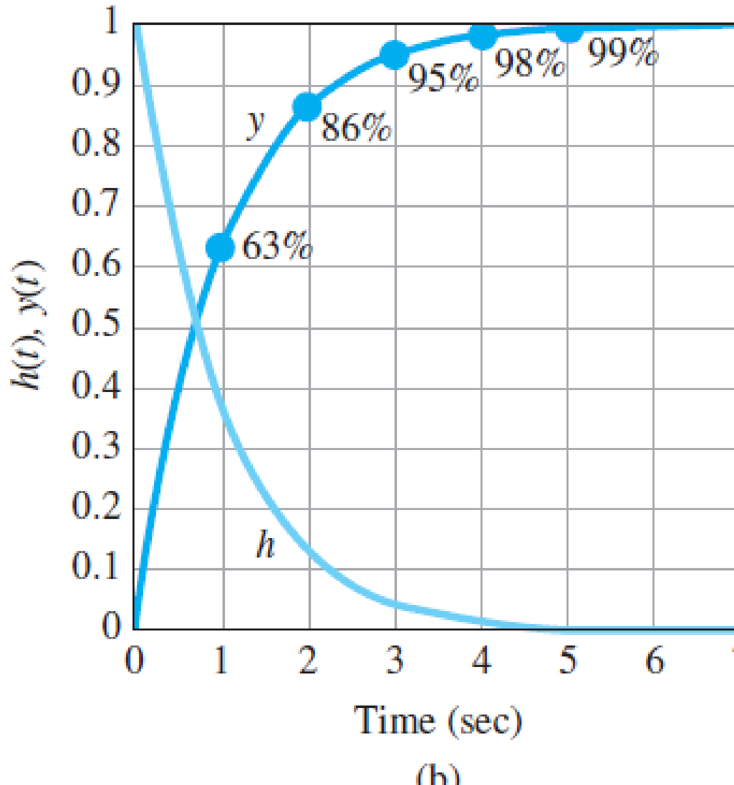
$$H(s) = \frac{1}{s+a} \quad \Rightarrow \quad h(t) = e^{-at}\,1(t)$$

- Impulse Response: $\Rightarrow \quad y(t) = e^{-at}\,1(t)$

- Step Response: $\Rightarrow \quad y(t) = \frac{1}{a}(1 - e^{-at})\,1(t)$



(a)

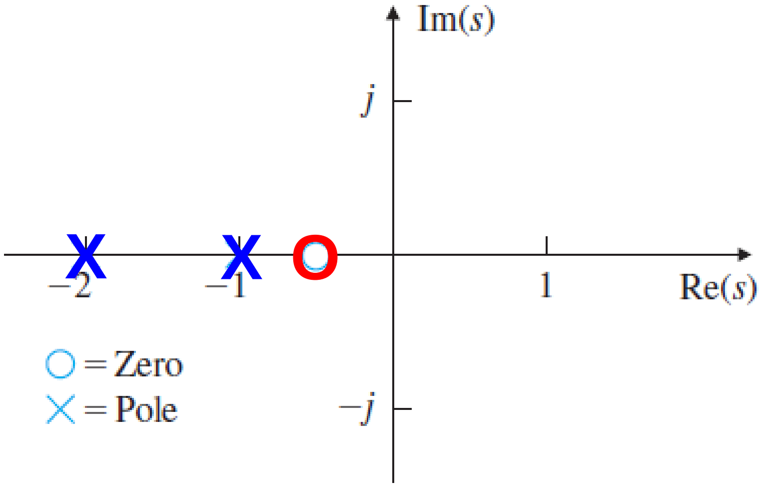


(b)

- Example 3.25: Response vs Pole Locations, Real Roots

$$H(s) = \frac{2s+1}{s^2+3s+2}$$

- poles as crosses **(x)** and zeros as circles **(o)**

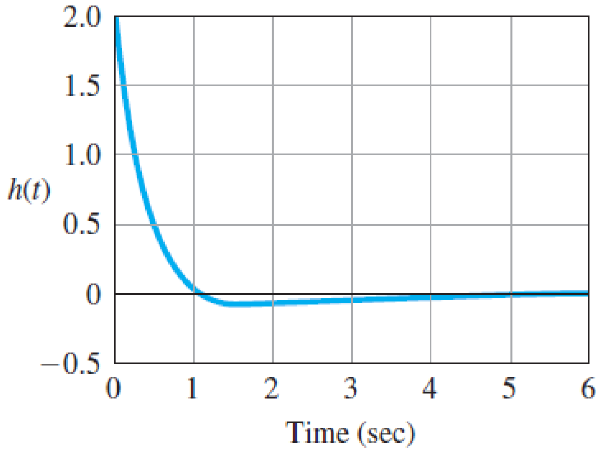


- a(s) = s^2 + 3s + 2 = 0

  Poles: s = -1, -2

- b(s) = 2s + 1 = 0

  Zeros: s = -1/2

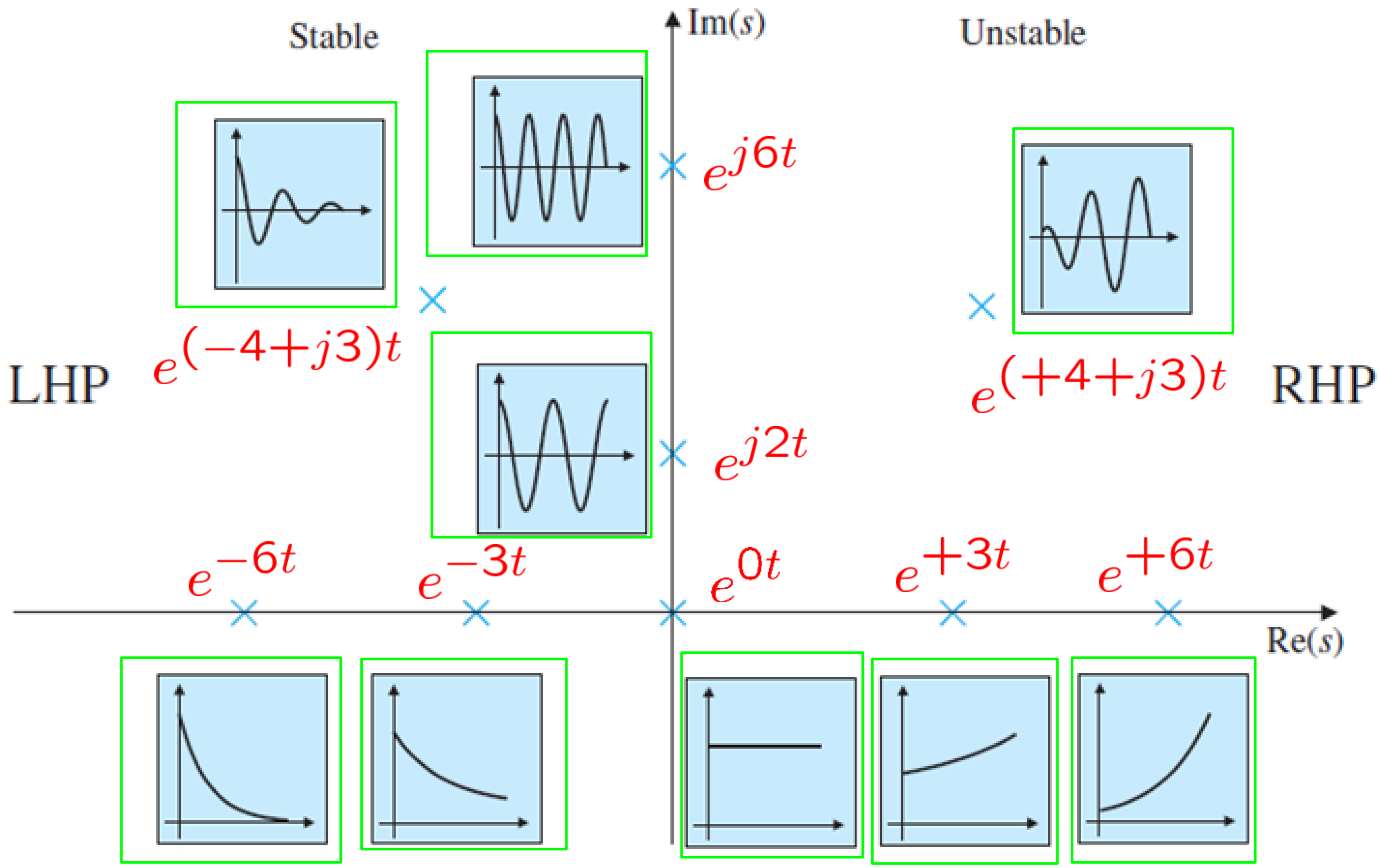


$$H(s) = -\frac{1}{s+1} + \frac{3}{s+2}$$

$$\Rightarrow h(t) = (-e^{-t} + 3e^{-2t})\,1(t)$$

Effect of Pole Locations

- Time functions associated with points in the s-plane
  (LHP, left half-plane; RHP, right half-plane)



Complex Poles: Damping Ratio, Undamped Natural Frequency

- Complex Poles $s = -\sigma \pm j w_d$

$$a(s) = (s + \sigma - j w_d)(s + \sigma + j w_d) = (s + \sigma)^2 + {w_d}^2$$

$$H(s) = \frac{{w_n}^2}{s^2 + 2\,\zeta\, w_n\, s + {w_n}^2}$$

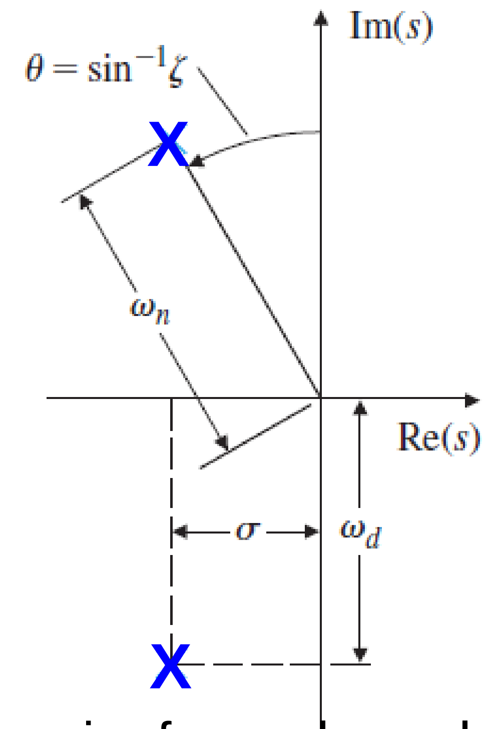
$$\sigma = w_n\,\zeta$$

$$w_d = w_n \sqrt{1 - \zeta^2}$$

- Damping Ratio: ζ
- Undamped Natural Frequency: ω_n

$$H(s) = \frac{{w_n}^2}{(s + \zeta w_n)^2 + {w_n}^2(1 - \zeta^2)}$$

$$h(t) = \frac{w_n}{\sqrt{1 - \zeta^2}}\, e^{-\sigma t}\, (\sin w_d t)\, 1(t)$$



- s-plane plot for a pair of complex poles

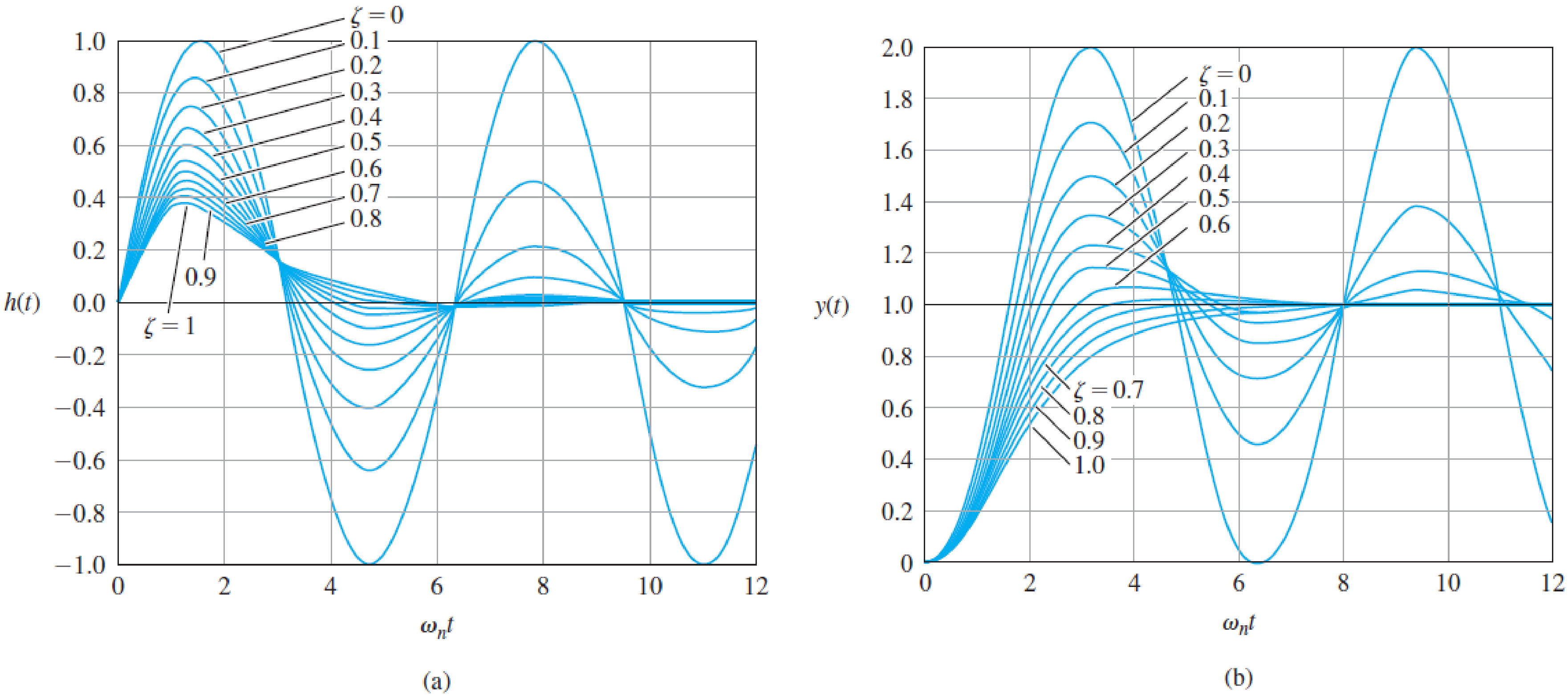
Complex Poles: Damping Ratio, Undamped Natural Frequency

$$H(s) = \frac{w_n{}^2}{(s+\zeta w_n)^2 + w_n{}^2(1-\zeta^2)} \qquad \sigma = w_n \zeta$$

$$h(t) = \frac{w_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} (\sin w_d t)\, 1(t) \qquad w_d = w_n \sqrt{1-\zeta^2}$$

- Responses of second-order systems versus ζ :

(a) Impulse Responses (b) Step Responses



Complex Poles: Damping Ratio, Undamped Natural Frequency

$$H(s) = \frac{{w_n}^2}{(s+\zeta w_n)^2 + {w_n}^2(1-\zeta^2)} \qquad \sigma = w_n\,\zeta$$

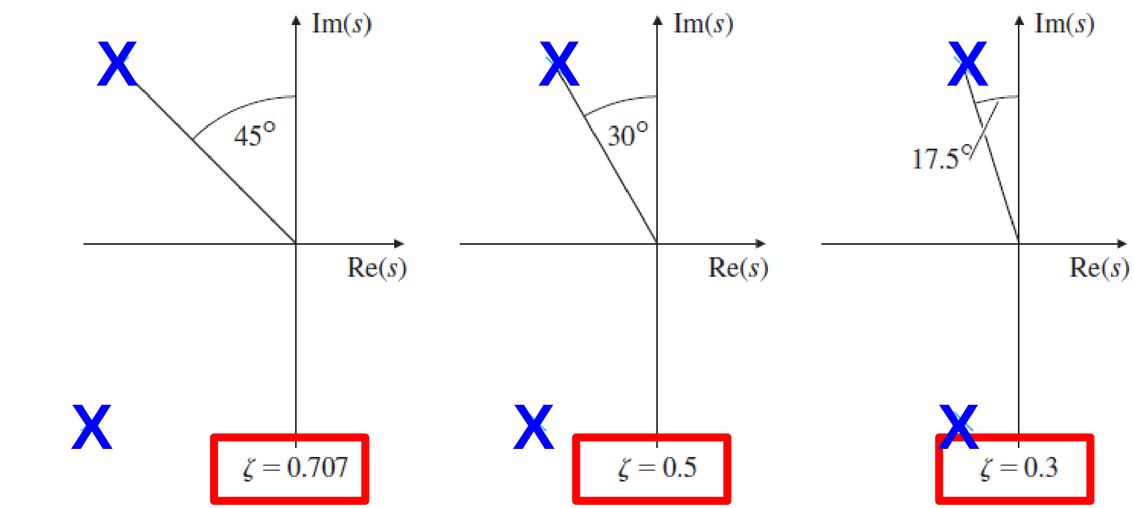
$$h(t) = \frac{w_n}{\sqrt{1-\zeta^2}}\, e^{-\sigma t}\,(\sin w_d t)\; 1(t) \qquad w_d = w_n\sqrt{1-\zeta^2}$$

```
s = tf( 's' )

wn = 1;
zeta = 0.3;
sysH = wn^2/( s^2 + 2*zeta*wn*s + wn^2 );

t = 0:0.01:10;
y1 = step( sysH, t );
y2 = impulse( sysH, t );
```

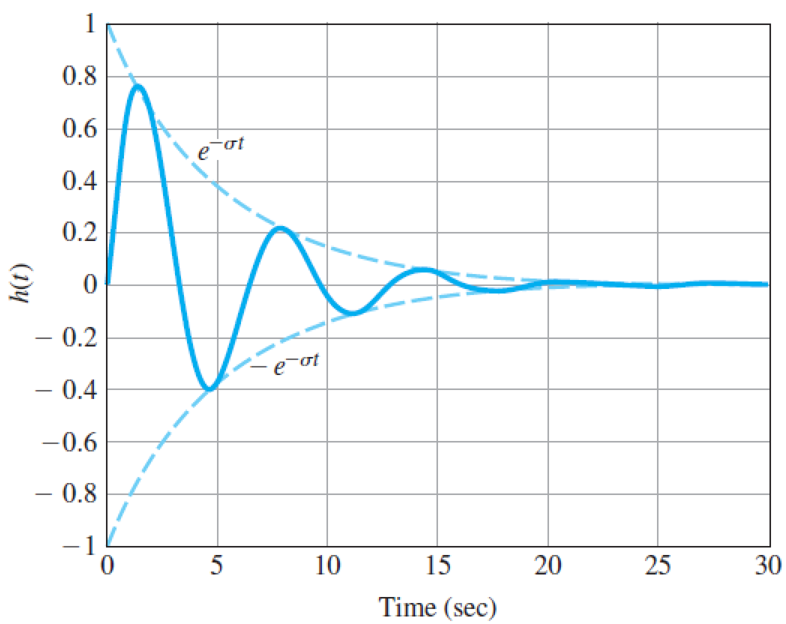
```
figure(1)

subplot(2,1,1)
plot( t, y1 )
grid
axis( [ 0, 10, -0.1, 2] )
title('step');
hold on;

subplot(2,1,2)
plot( t, y2 )
grid
axis( [ 0, 10, -2, 2] )
title('impluse');
hold on;
```

Complex Poles: Damping Ratio, Undamped Natural Frequency

- Stability depends on whether natural response grows or decays



- Pole locations corresponding to three values of ζ



$$h(t) = \frac{w_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} (\sin w_d t)\, 1(t)$$

- Second-order system response with an exponential envelope bound

- Example 3.26: Oscillatory Time Response

$$H(s) = \frac{2s+1}{s^2+2s+5} = \frac{2s+1}{(s+1)^2+2^2}$$

$$w_n{}^2 = 5 \qquad w_n = \sqrt{5} = 2.24 \text{ rad/sec}$$

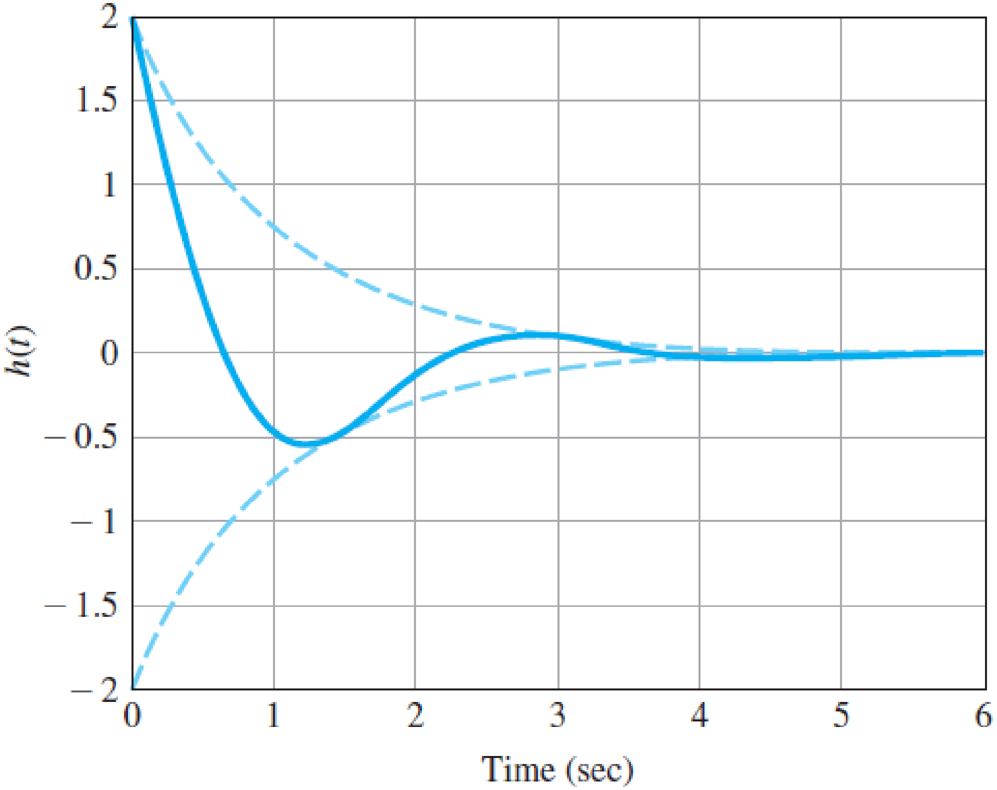
$$2\,\zeta\,w_n = 2 \qquad \zeta = \frac{1}{\sqrt{5}} = 0.447$$

$$H(s) = 2\,\frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

$$\Rightarrow\ h(t) = [\,2\,e^{-t}\,(\cos 2t) - \frac{1}{2}\,e^{-t}\,(\sin 2t)\,]\,1(t)$$

Example 3.26: Oscillatory Time Response

```
s = tf( 's' )

sysH = ( 2*s + 1 )/( s^2+2*s+5 );

t = 0:0.1:6;
y = impulse( sysH, t );

plot( t, y )
grid
hold on;
plot( t, 2*exp(-t), ':' )
plot( t, -2*exp(-t), ':' )
```



- system response