

Spring 2021

控制系統  
Control Systems

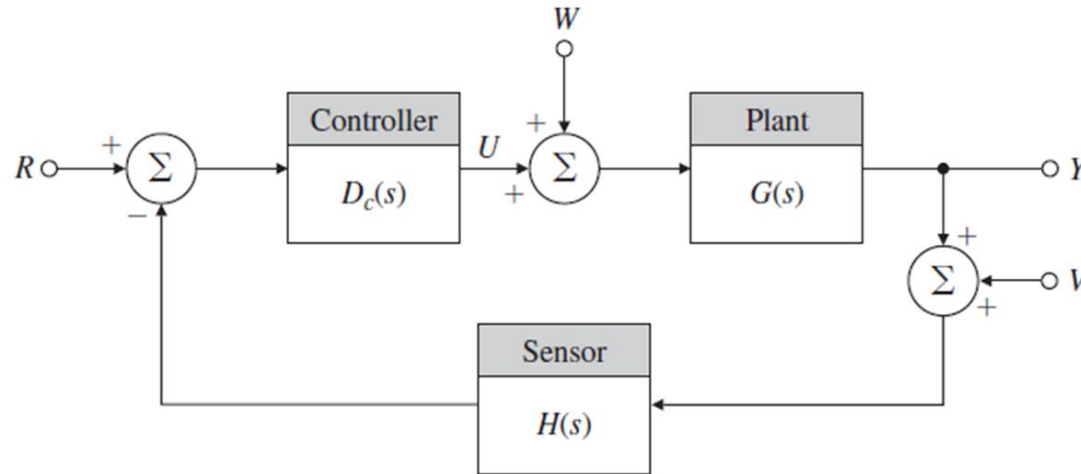
Unit 5A  
Root Locus of a Basic Feedback System

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- Basic Feedback System:



- Closed-Loop Transfer Function:

$$\mathcal{T}(s) = \frac{Y(s)}{R(s)} = \frac{D_c(s) G(s)}{1 + D_c(s) G(s) H(s)}$$

- The Characteristic Equation:

$$1 + D_c(s) G(s) H(s) = 0 \quad \Rightarrow \quad 1 + K \frac{b(s)}{a(s)} = 0$$

- The roots are the poles of the transfer function  $\Rightarrow a(s) + K b(s) = 0$

## Example 0

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0 \quad \Rightarrow a(s) + K b(s) = 0$$

$$\Rightarrow 1 + K \frac{s+1}{s^2+4s+8} = 0 \quad \Rightarrow s^2+4s+8 + K(s+1) = 0$$

$$\Rightarrow s^2 + (4+K)s + (8+K) = 0$$

■ **K**      **roots**

■ **0**      -2      + j(2)

-2      - j(2)

■ **1**      -2.5      + j(1.66)

-2.5      - j(1.66)

■ **2**      -3      + j(1)

-3      - j(1)

■ **2.4**      -3.2      + j(0.4)

-3.2      - j(0.4)

■ **2.47**      -3.24      -3.24

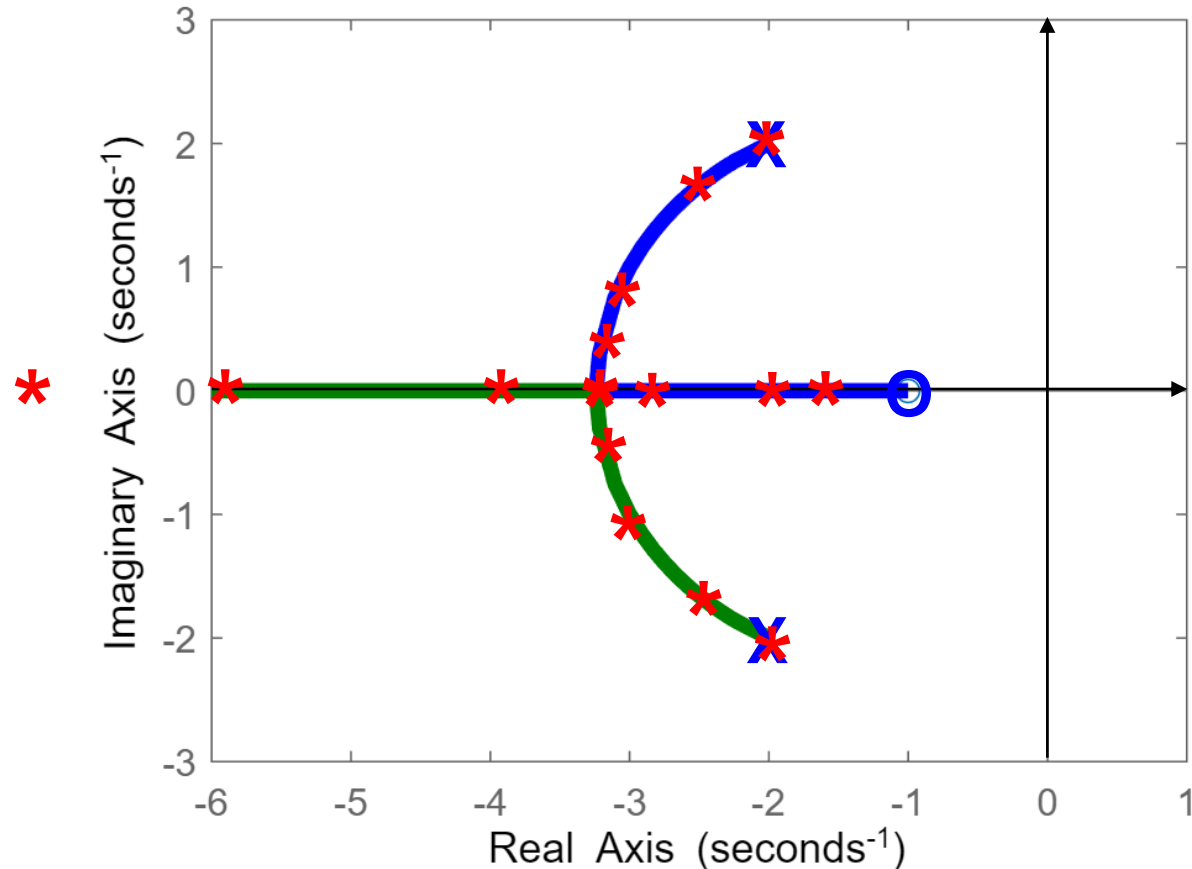
■ **2.6**      -3.84      -2.76

■ **3**      -4.62      -2.38

■ **4**      -6      -2

■ **5**      -7.19      -1.81

Root Locus vs. K



- Root Locus:

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0 \quad \Rightarrow a(s) + K b(s) = 0$$

- $K$ : From 0 to infinity

- The plot of all possible roots

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

$$= (s - p_1) (s - p_2) \dots (s - p_{n-1}) (s - p_n) = \prod_{i=1}^n (s - p_i)$$

$$b(s) = s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_{m-1} s + b_m$$

$$= (s - z_1) (s - z_2) \dots (s - z_{m-1}) (s - z_m) = \prod_{i=1}^m (s - z_i)$$

- For simplicity, assume that  $n \geq m$ ,

- That is, it is a rational transfer function

$$\Rightarrow a(s) + K b(s) = (s - r_1) (s - r_2) \dots (s - r_{n-1}) (s - r_n)$$

- The **Root-Locus Form or Evan Form** of a Characteristic Equation:

$$1 + D_c(s) G(s) H(s) = 0$$

$$\Rightarrow 1 + K L(s) = 0$$

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0$$

$$\Rightarrow a(s) + K b(s) = 0$$

$$\Rightarrow L(s) = -\frac{1}{K}$$

- The **Root-Locus Method**:

a method for inferring **dynamic properties** of the closed-loop systems  
as **K** changes

# Examples

## Example 5.1: Root Locus of a Motor Position Control

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{Y(s)}{U(s)} = G(s) = \frac{A}{s(s+c)}$$

$$D_c(s) = H(s) = 1, \quad c = 1$$

$$\Rightarrow L(s) = \frac{b(s)}{a(s)} = \frac{1}{s(s+1)} \Rightarrow \begin{cases} m = 0 & z_i = \{ \text{empty} \} \\ n = 2 & p_i = \{0, -1\} \end{cases}$$

$$\Rightarrow K = A$$

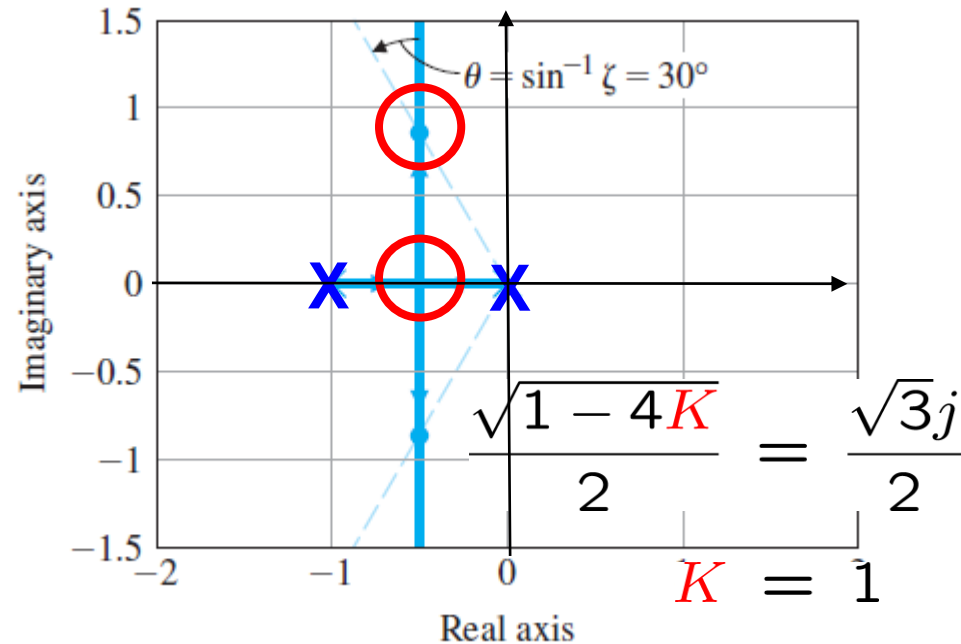
$$\Rightarrow a(s) + K b(s) = s^2 + s + K = 0$$

$$\Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{1 - 4K}}{2}$$

$$\Rightarrow 1 - 4K = 0$$

$$\Rightarrow r_{1,2} = -\frac{1}{2}, -\frac{1}{2}$$

$$\Rightarrow K = \frac{1}{4} \quad \blacksquare \text{ Breakaway point}$$



# Examples

## Example 5.2: Root Locus w.r.t. a Plant Open-Loop Pole

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{Y(s)}{U(s)} = G(s) = \frac{1}{s(s+c)}$$

$$1 + G(s) = 1 + \frac{1}{s(s+c)} \Rightarrow s^2 + cs + 1 = 0$$

$$\Rightarrow L(s) = \frac{b(s)}{a(s)} = \frac{s}{s^2 + 1} \Rightarrow \begin{cases} m = 1 & z_i = \{0\} \\ n = 2 & p_i = \{+j, -j\} \end{cases}$$

$$\Rightarrow K = c$$

$$\Rightarrow 1 + c \frac{s}{s^2 + 1} = 0$$

$$\Rightarrow r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4}}{2}$$

$$\Rightarrow c = 2 \quad \blacksquare \text{ Break-in point}$$

$$\Rightarrow r_{1,2} = -1, -1$$

