Spring 2021

控制系統
Control Systems

Unit 50 & 60
Root Locus (s-Domain) and Bode Plot (w-Domain)

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NTU-EE

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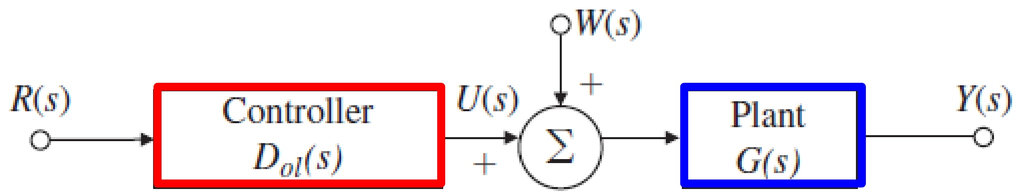
Open-Loop and Closed-Loop Systems

- Open-loop system showing
  reference, R, control, U, disturbance, W, and output Y

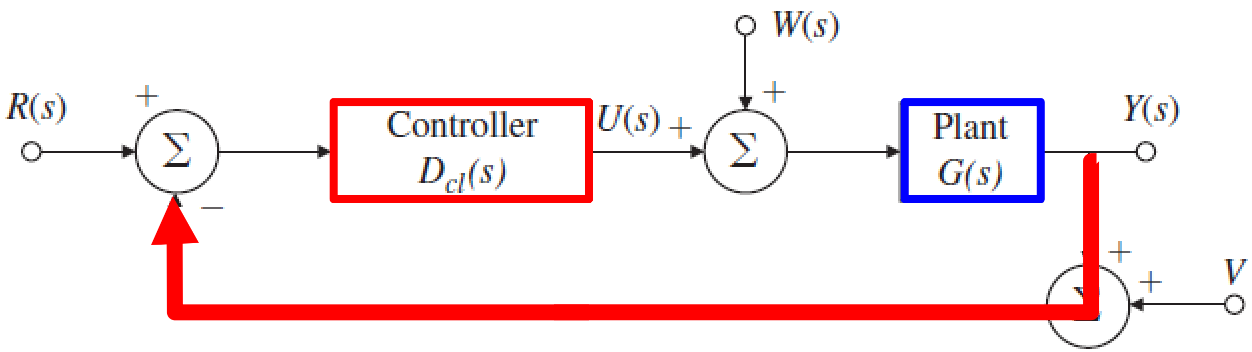
$$G = \frac{b(s)}{a(s)}$$

$$D = \frac{c(s)}{d(s)}$$



$$T_{ol} = \frac{Y_{ol}}{R} = G\ D_{ol} = \frac{b(s)}{a(s)}\ \frac{c(s)}{d(s)}$$

$$T_{cl} = \frac{Y_{cl}}{R} = \frac{G\ D_{cl}}{1 + G\ D_{cl}} = \frac{b(s)\ c(s)}{1 + a(s)\ d(s)}$$

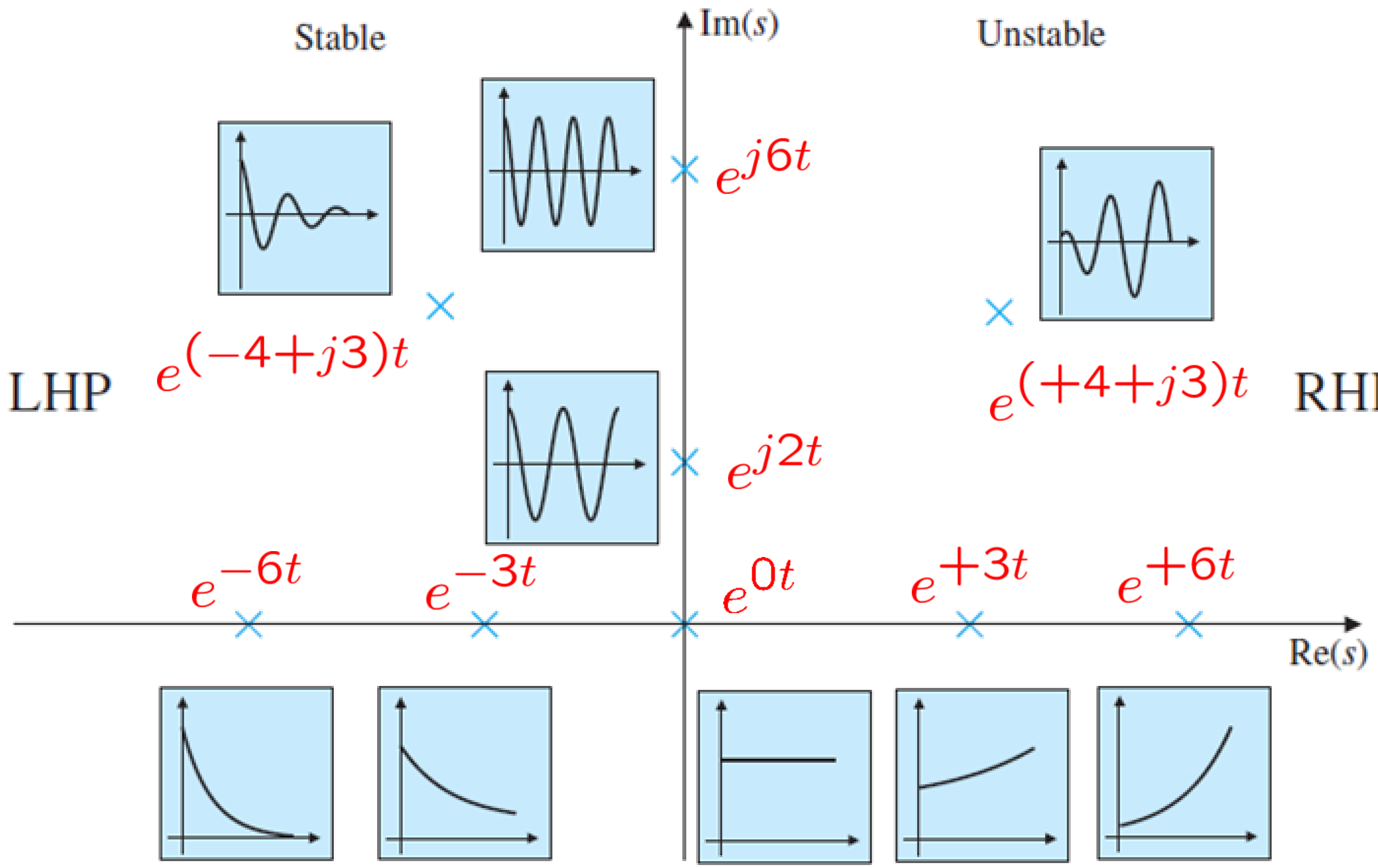


- Poles
  of Transfer Function
- Roots
  of Characteristic Equation

- Closed-loop system showing
  reference, R, control, U, disturbance, W, output, Y, and sensor noise, V

Effect of Pole Locations

- Time functions associated with points in the s-plane
  (LHP, left half-plane; RHP, right half-plane)



- Overshoot M_p and Peak time t_p

$$Y(s) = H(s)\,\frac{1}{s}$$

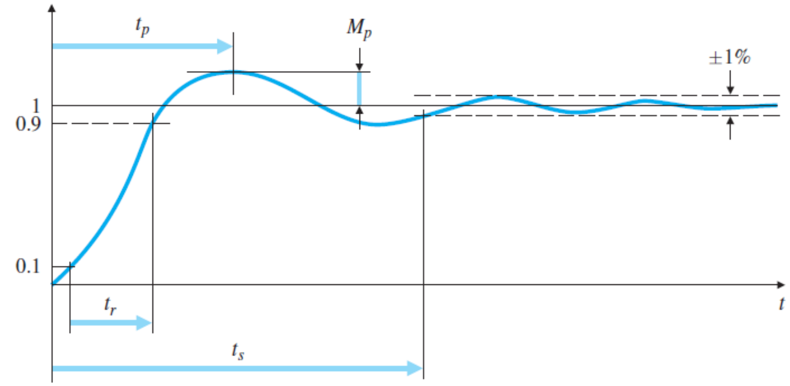
$$H(s) = \frac{w_n^2}{s^2 + 2\,\zeta\,w_n\,s + w_n^2}$$

$$\sigma = w_n\,\zeta$$

$$w_d = w_n\sqrt{1-\zeta^2}$$

$$\Rightarrow\ y(t) = 1 - e^{-\sigma t}\left(\cos w_d t + \frac{\sigma}{w_d}\sin w_d t\right)$$

$$\Rightarrow\ y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(w_d t - \beta)$$



$$A\sin(\alpha) + B\cos(\beta) = C\cos(\alpha-\beta)$$

$$C = \sqrt{A^2+B^2} = \frac{1}{\sqrt{1-\zeta^2}}$$

$$\beta = \tan^{-1}\left(\frac{A}{B}\right)$$

$$= \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = \sin^{-1}(\zeta)$$

$$\Rightarrow\ M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\Rightarrow\ t_r \cong \frac{1.8}{w_n}$$

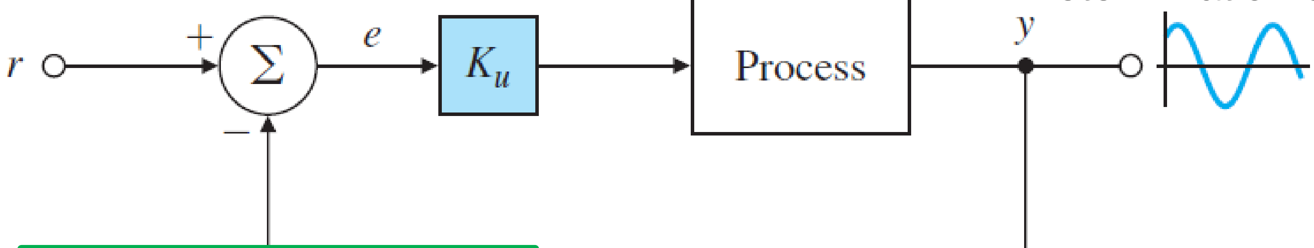
$$\Rightarrow\ t_s = \frac{4.6}{\zeta w_n} = \frac{4.6}{\sigma}$$

Ziegler–Nichols Tuning of the PID Controller

- **Method 2: Ultimate Sensitivity Method:**

  Based on evaluating the amplitude and frequency of the oscillations of the system at the limit of stability rather than on taking a step response.

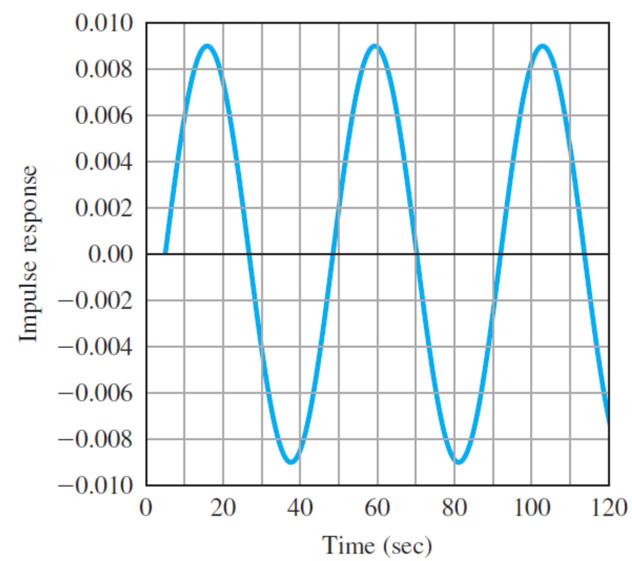
  - Determination of ultimate gain and period



- K_u : Ultimate Gain
- P_u : Ultimate Period

Ziegler-Nichols Tuning for the Regulator $D_c(s) = k_P(1 + 1/T_I s + T_D s)$, Based on the Ultimate Sensitivity Method

Type of Controller	Optimum Gain
P	$k_P = 0.5K_u$
PI	$k_P = 0.45K_u$, $T_I = \frac{P_u}{1.2}$
PID	$k_P = 0.6K_u$, $T_I = 0.5P_u$, $T_D = 0.125P_u$



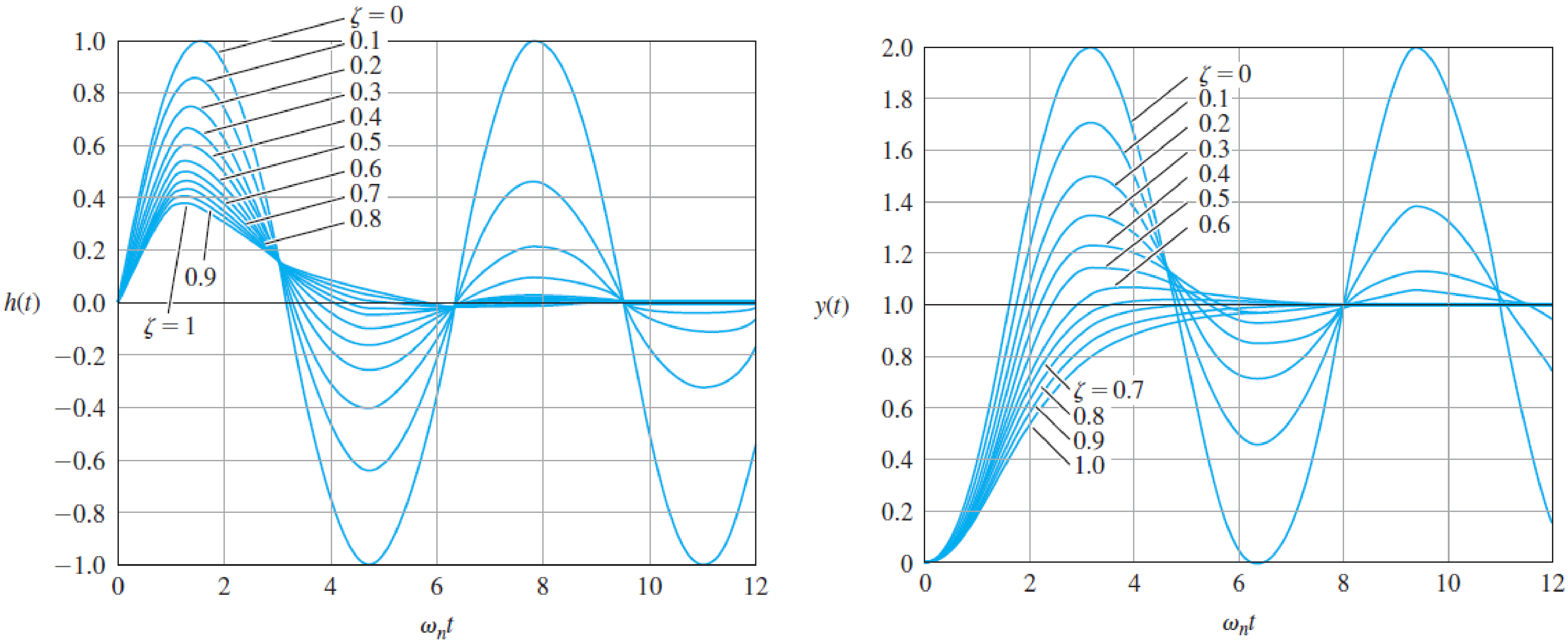
Complex Poles: Damping Ratio, Undamped Natural Frequency

$$H(s) = \frac{w_n^2}{(s+\zeta w_n)^2 + w_n^2(1-\zeta^2)} \qquad \sigma = w_n \zeta$$

$$h(t) = \frac{w_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} (\sin w_d t)\, 1(t) \qquad w_d = w_n \sqrt{1-\zeta^2}$$

- Responses of second-order systems versus ζ :

(a) Impulse Responses (b) Step Responses



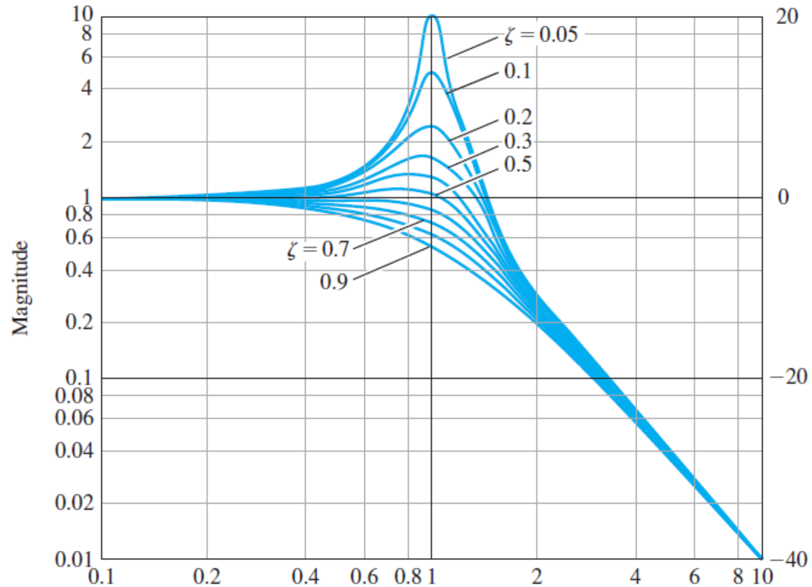
(a) (b)

Frequency Response and Bode Plot

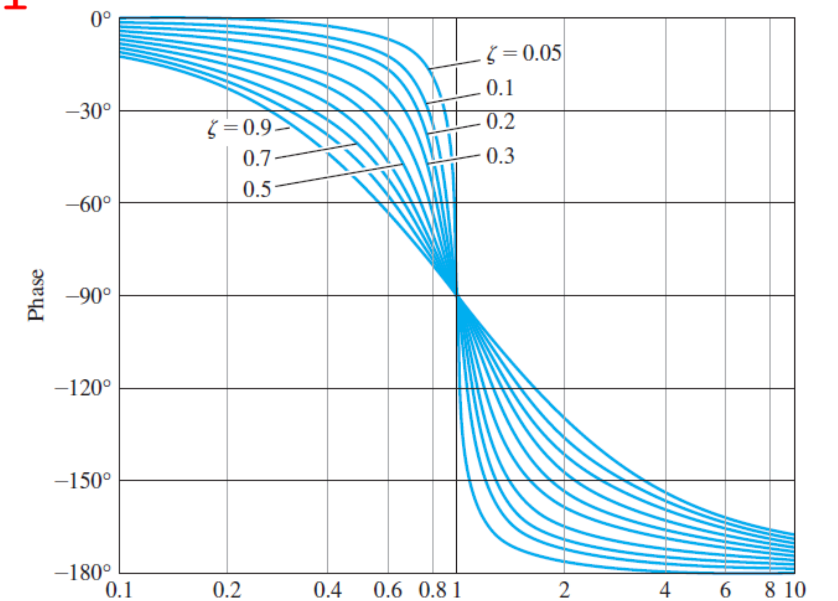
$$|G(s = jw)|$$

$$G(s) = \frac{1}{(s/w_n)^2 + 2\zeta(s/w_n) + 1}$$

$$\measuredangle G(s = jw)$$



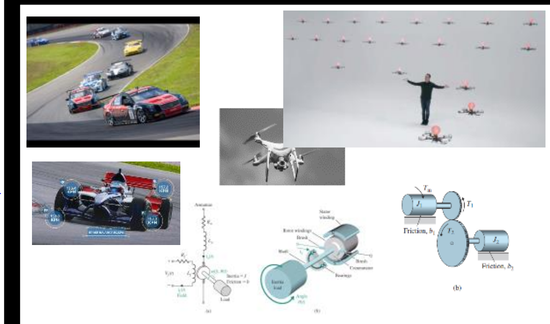
w/w_n



w/w_n

Plant, Input, Output, Action, Goal

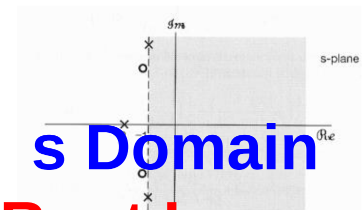
Plant (P)



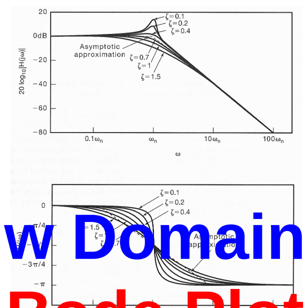
Ref (r) → **Input (u)** → Plant → **Output (y)**

$$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} - 3\,y(t) = 5\,u(t)$$

$$P(s) = \frac{Y(s)}{U(s)} = \frac{5}{s^2 + 2\,s - 3}$$



s Domain
Root Locus



w Domain
Bode Plot

Controller

1. **Model**
2. **Response**
3. **Analysis**
4. **Feedback**
5. **Control**

$$\frac{d^2y(t)}{dt^2} + 4\,\frac{dy(t)}{dt} + 3\,y(t) = 3\,r(t)$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{3}{s^2 + 4\,s + 3}$$

Unit 5

Root Locus

Your Jobs are To

- By Hand:
  - **Hand Writing in Exam (40%)**
- Use the 5 rules of Root Locus Method

  to roughly sketch the root locus of any transfer function

  by identifying these critical root locations
- Properly choose some roots

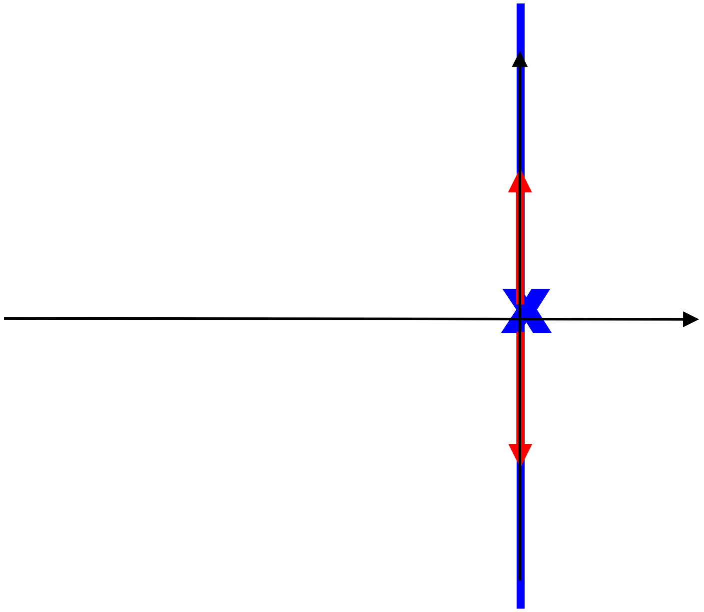
  between these critical root locations

- By Computer:
  - **Multiple Choice in Exam (60%)**
- Use Matlab codes

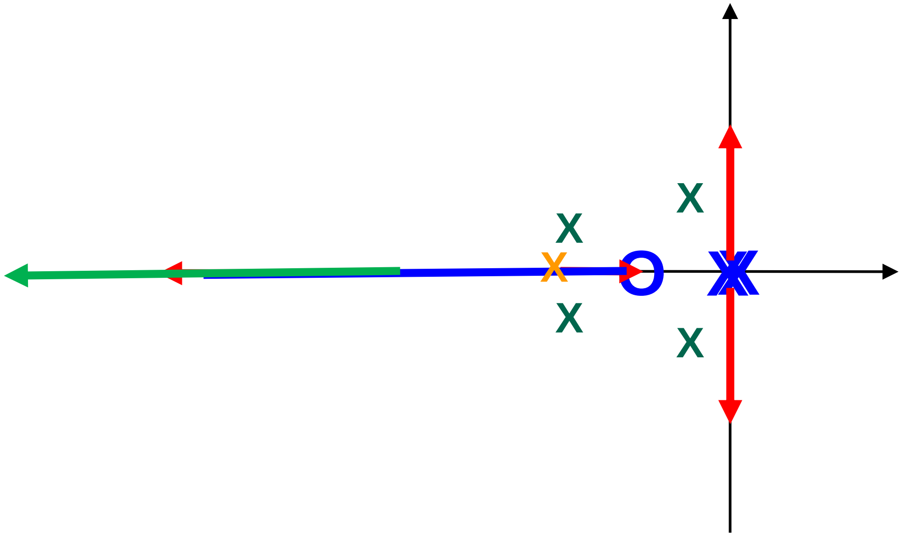
  to draw the exact root locus of any transfer function
- Design proper transfer function and

  select associated and reasonable gain value

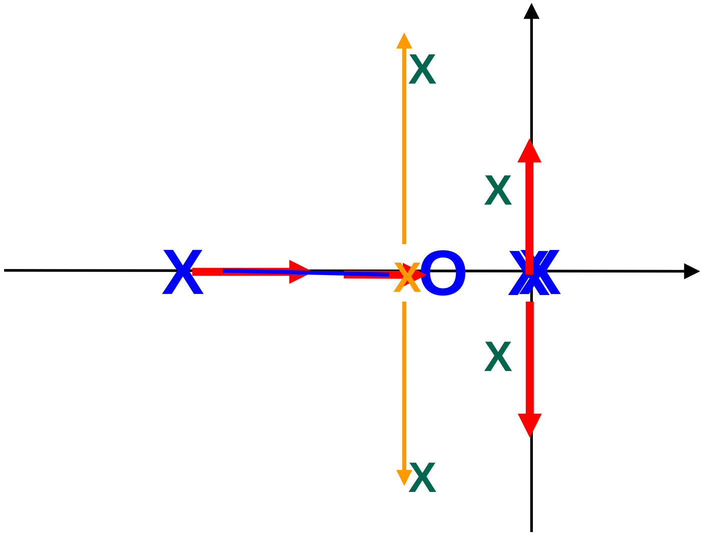
$$\Rightarrow 1 + K_P \frac{1}{s^2} = 0$$



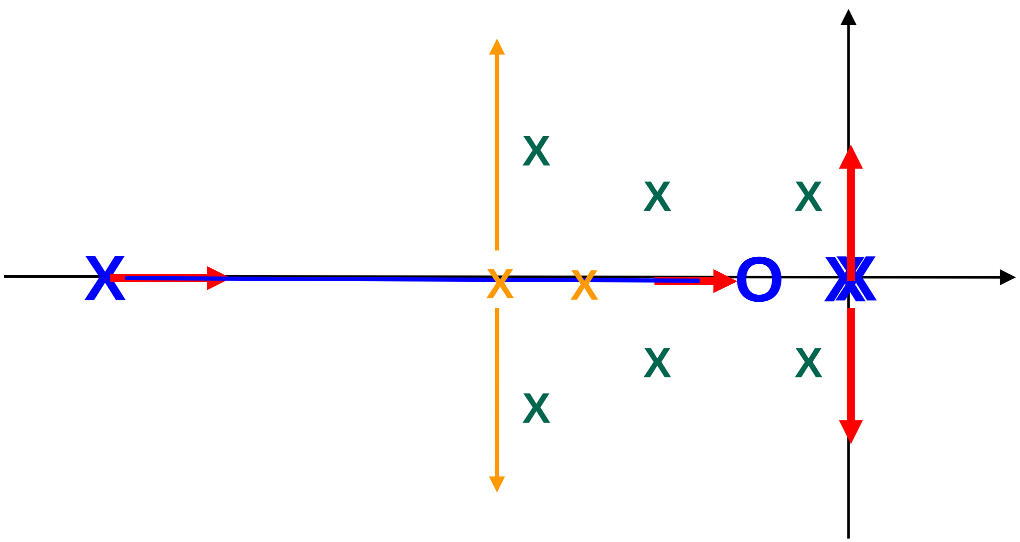
$$\Rightarrow 1 + K \frac{s+1}{s^2} = 0$$



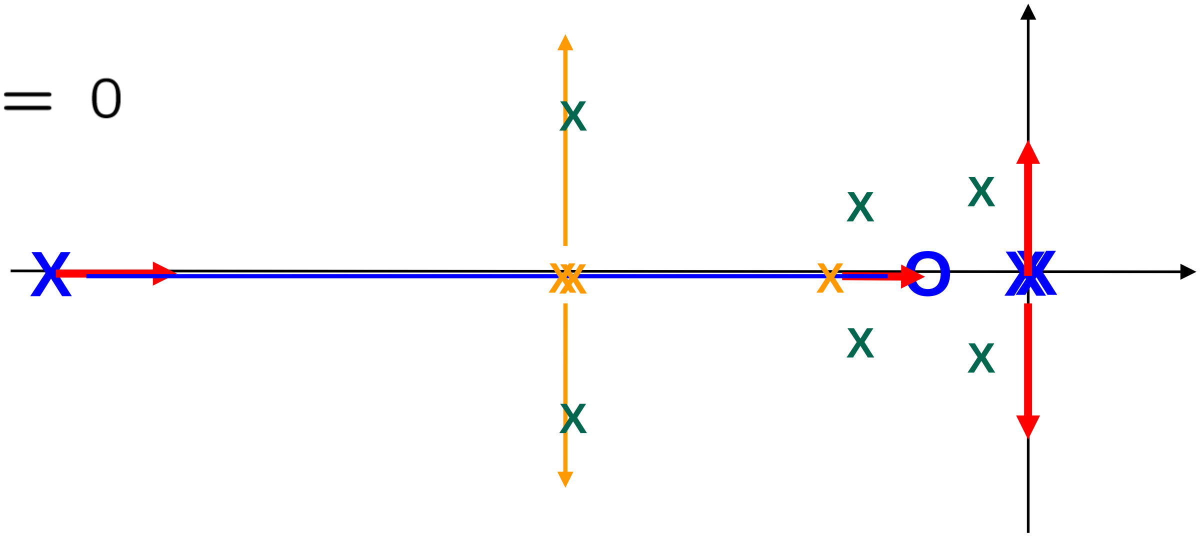
$$\Rightarrow 1 + K \frac{(s+1)}{s^2 (s+4)} = 0$$



$$\Rightarrow 1 + K \frac{(s+1)}{s^2 (s+9)} = 0$$

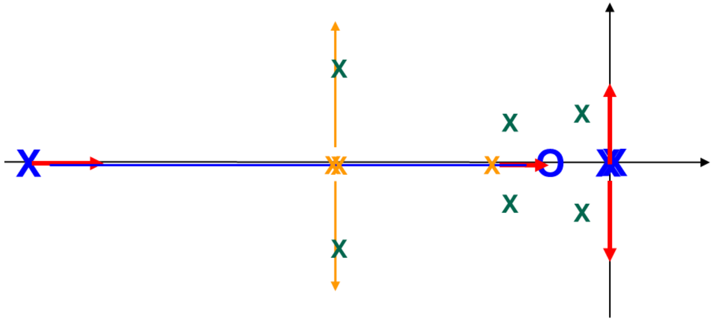


$$\Rightarrow \; 1 + K \frac{(s+1)}{s^2 (s+12)} = 0$$

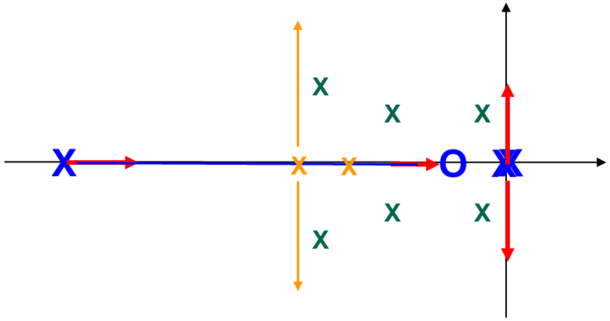


Your Jobs By Hand

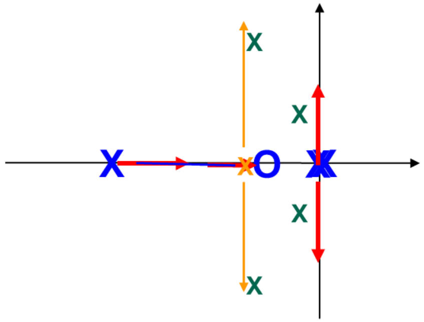
$$\Rightarrow \; 1 + K \frac{(s+1)}{s^2\,(s+12)} = 0$$



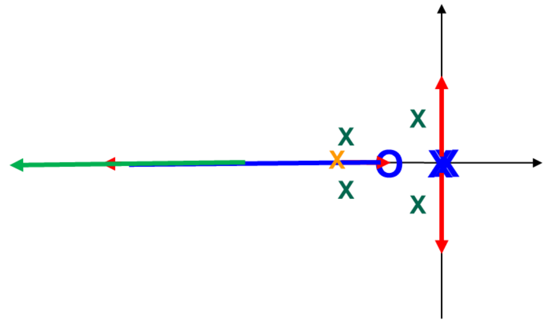
$$\Rightarrow \; 1 + K \frac{(s+1)}{s^2\,(s+9)} = 0$$



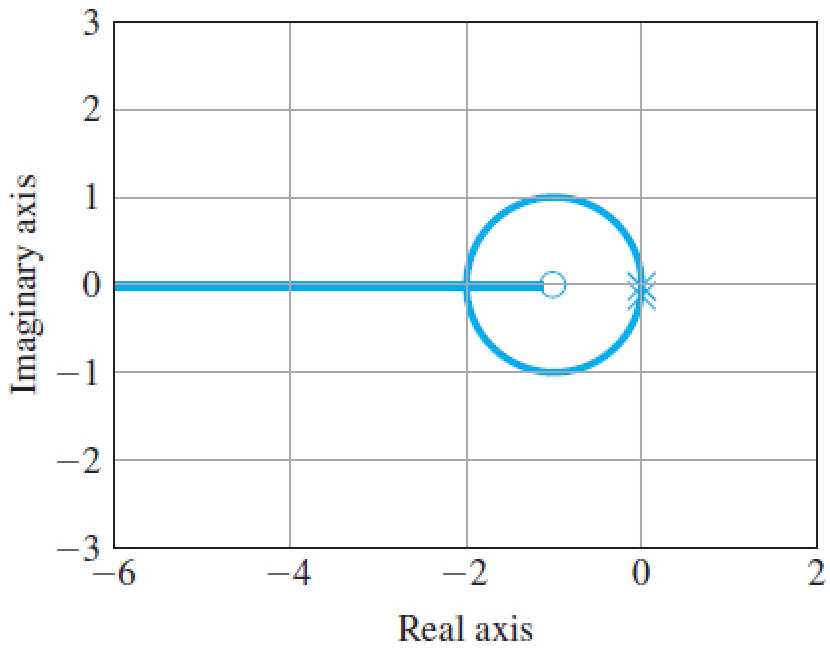
$$\Rightarrow \; 1 + K \frac{(s+1)}{s^2\,(s+4)} = 0$$



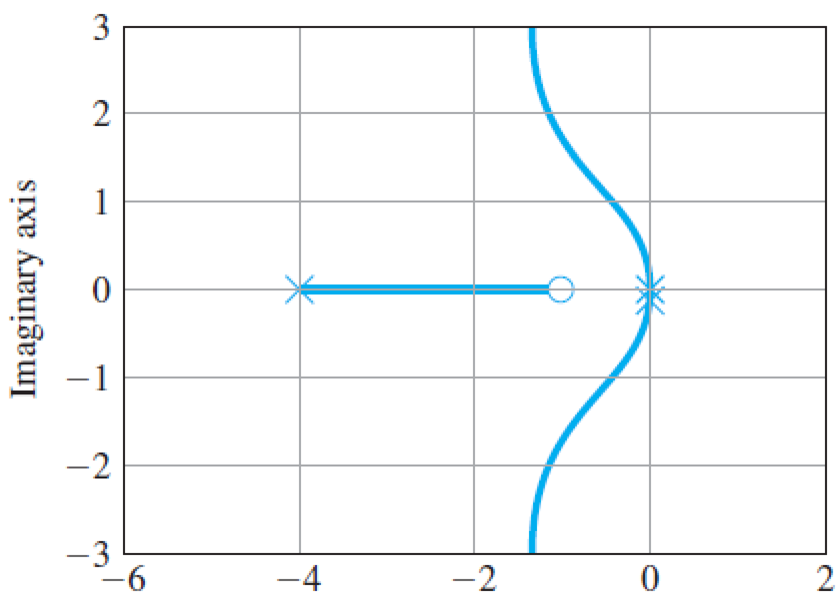
$$\Rightarrow \; 1 + K \frac{s+1}{s^2} = 0$$



$$\Rightarrow 1 + K \frac{s+1}{s^2} = 0$$



$$\Rightarrow 1 + K \frac{(s+1)}{s^2 (s+4)} = 0$$



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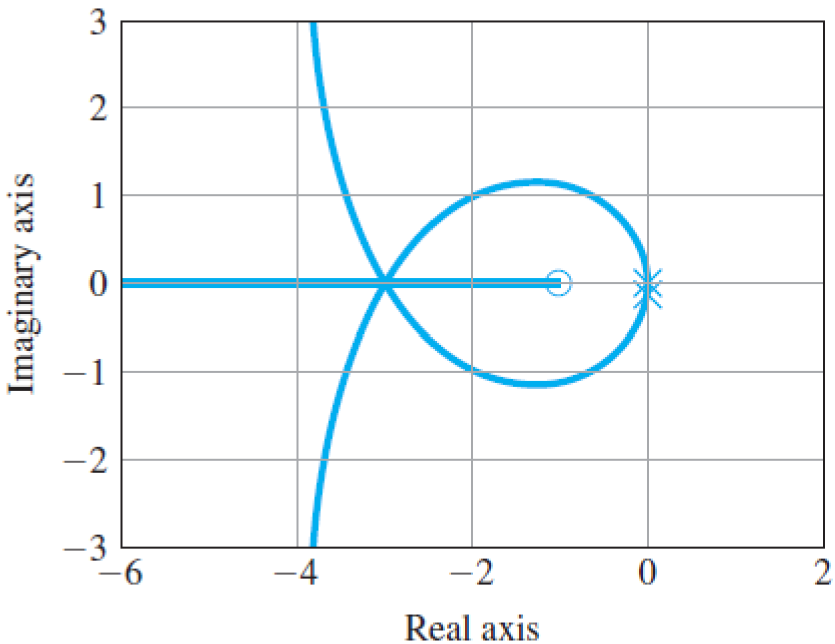
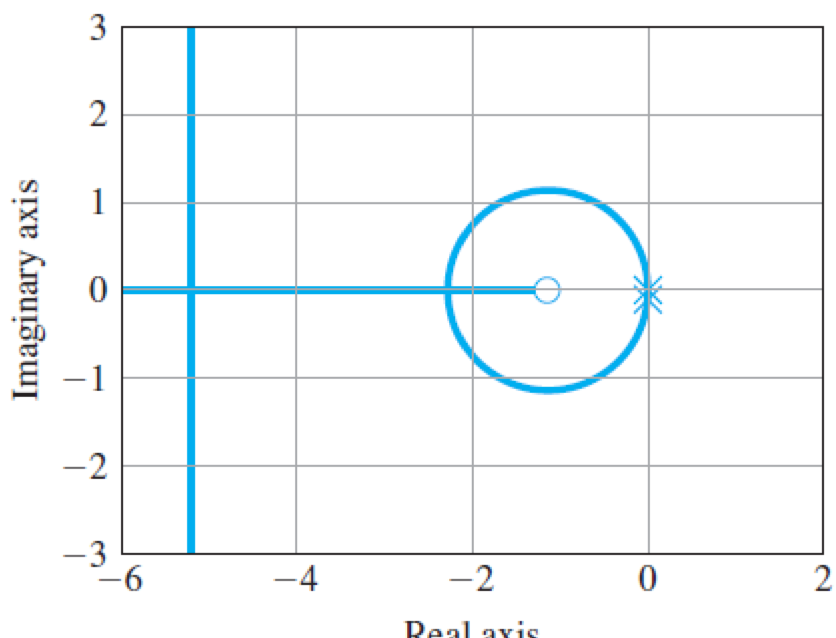
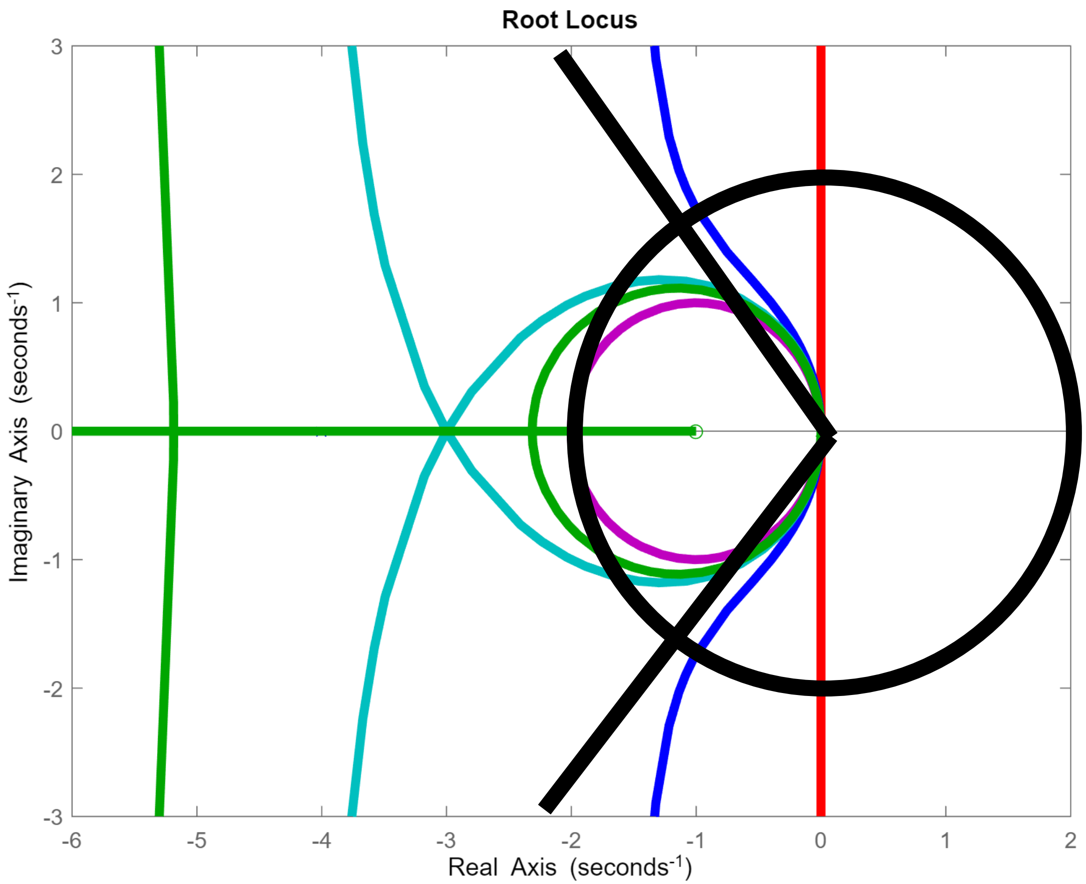
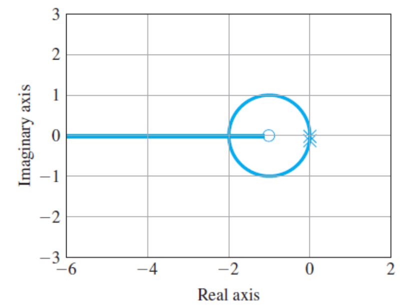
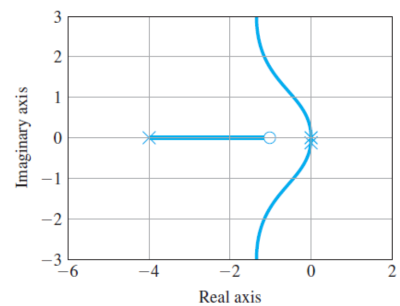


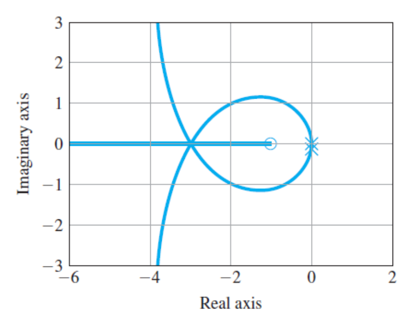
$$\Rightarrow \; 1 + K \frac{(s+1)}{s^2\,(s+12)} = 0$$

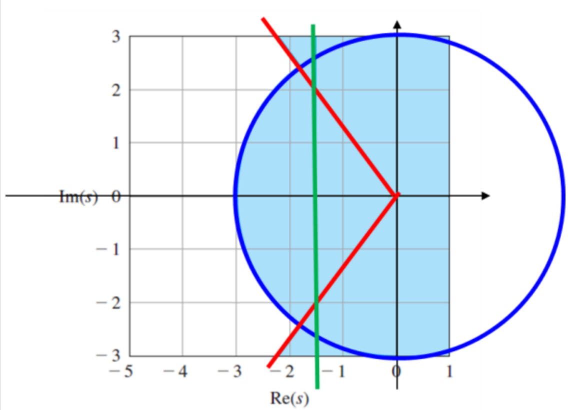


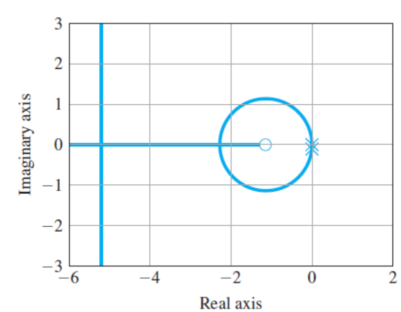










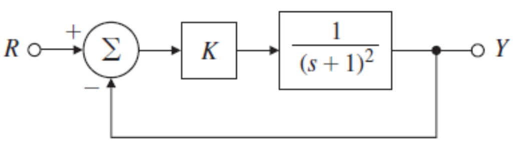


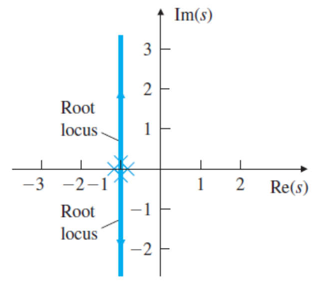
- Proper transfer function
- Reasonable gain values

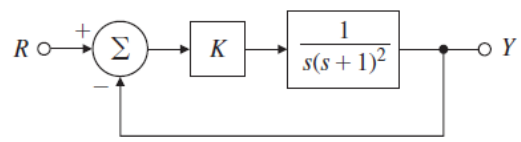
Unit 6

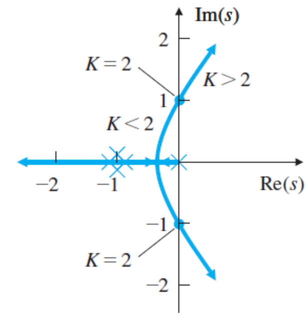
Bode Plot

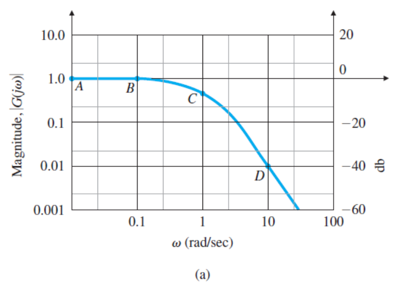
Key Ingredients

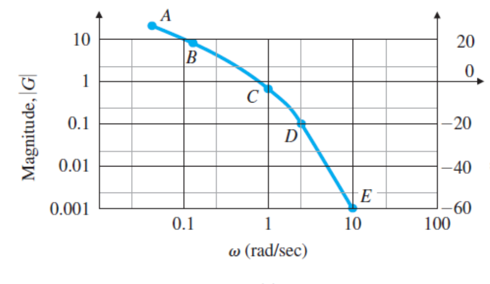


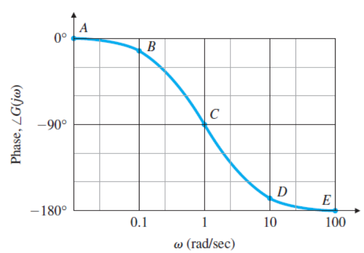


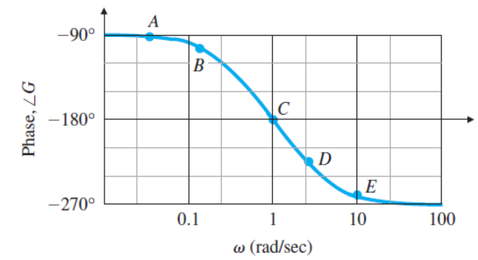


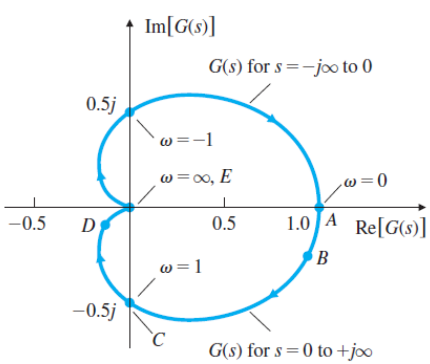


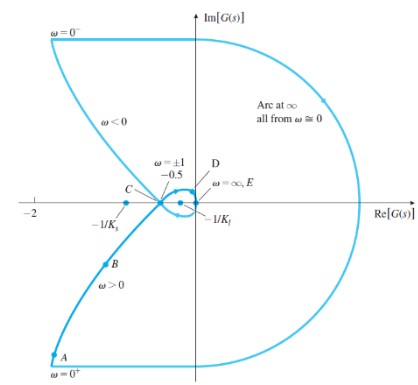












- Root Locus:
- Bode Plot:
- Nyquist Plot:

Your Jobs are To

- By Hand:
  - **Hand Writing in Exam (40%)**
- Use the Bode Plot Techniques

  to roughly sketch the magnitude and phase plots

  of any transfer function

  by identifying these critical frequency locations

- By Computer:
  - **Multiple Choice in Exam (60%)**
- Use Matlab codes

  to draw the exact Bode Plot, Nyquist Plot

  of any transfer function

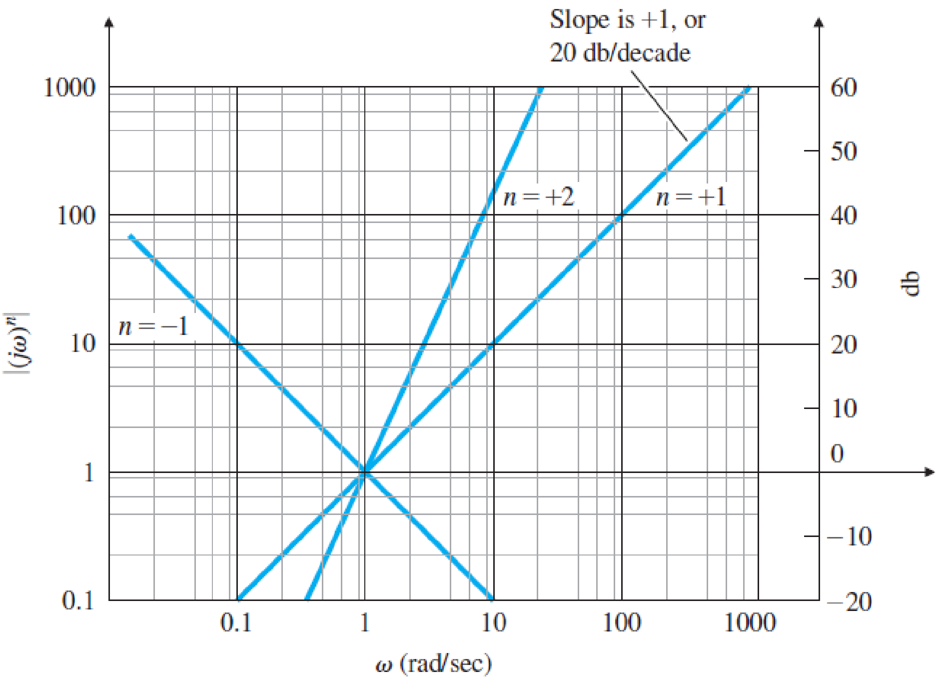
- Design proper transfer function and

  select associated and reasonable gain value

- Class 1: Singularities at the origin $K_0 \, (jw)^n$

$$\log K_0 |(jw)^n| = \log K_0 + n \log |jw|$$

$$\angle K_0 \, (jw)^n = n \times 90^0$$

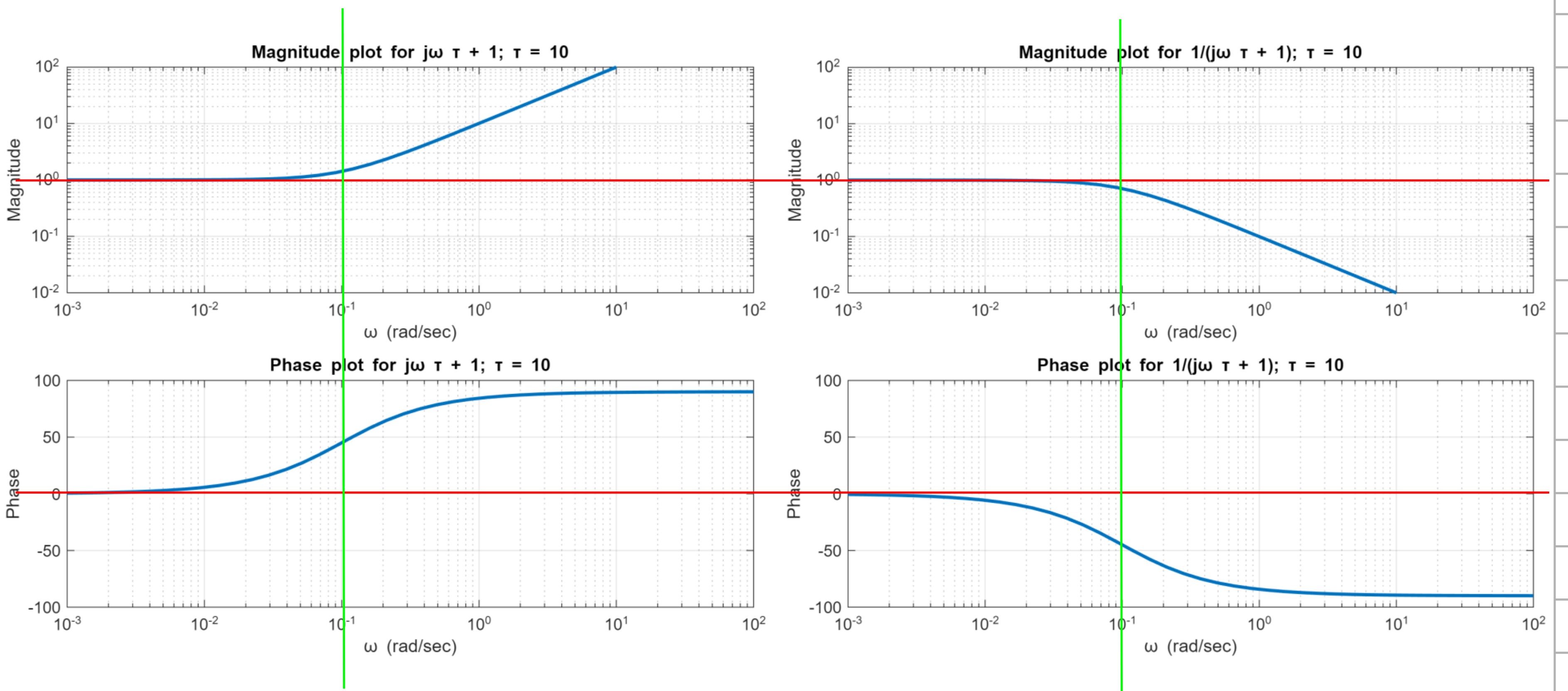


- Class 2: Frist-order term

$$(jw\tau + 1)^{\pm 1}$$

$$G(s) = 10\,s + 1$$

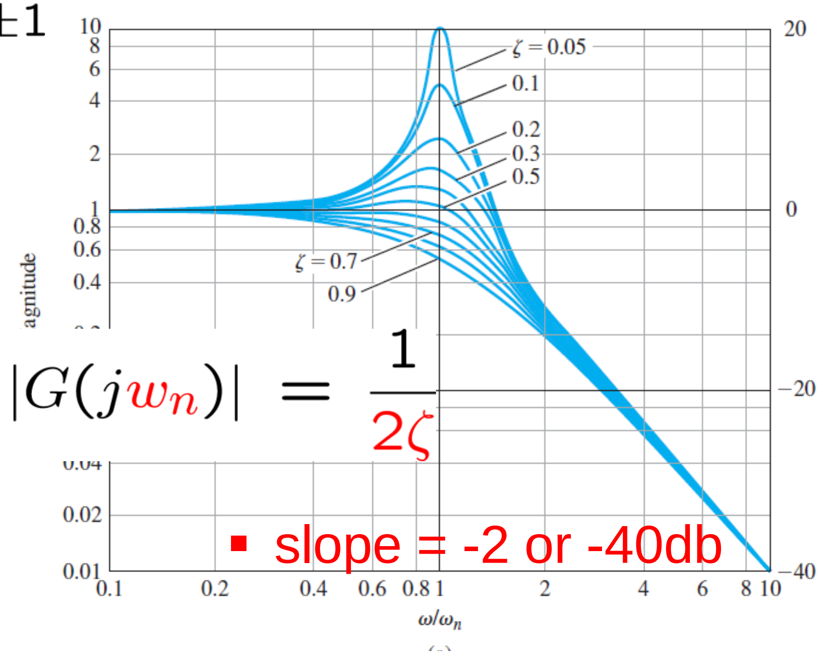
$$G(s) = \frac{1}{10\,s + 1}$$



Your Jobs by Hands and Pens

- Class 3: $\left[\left(\frac{jw}{w_n} \right)^2 + 2\,\zeta\, \frac{jw}{w_n} + 1 \right]^{\pm 1}$

$$|\,G(s = jw)\,|$$

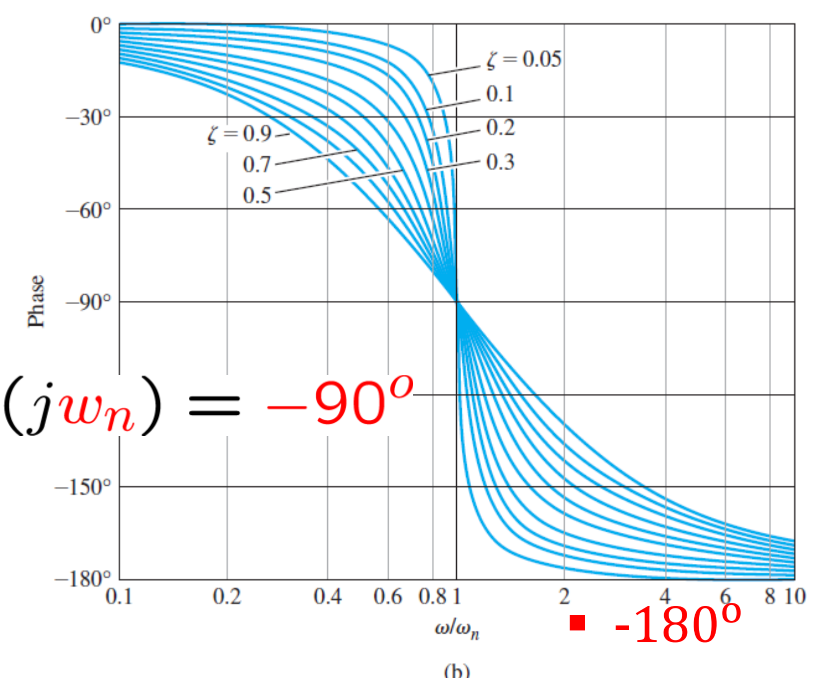


$$|G(jw_n)| = \frac{1}{2\zeta}$$

- slope = -2 or -40db

$$G(s) = \frac{1}{(s/w_n)^2 + 2\zeta(s/w_n) + 1}$$

$$\angle G(s = jw)$$



$$\angle G(jw_n) = -90^o$$

- -180^o

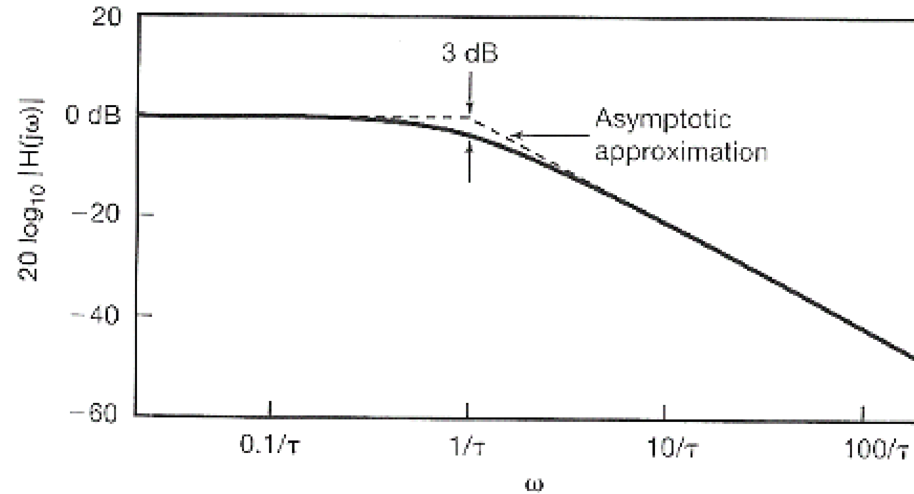
Your Jobs by Hands and Pens

- First-Order CT Systems:

$$H(jw) = \frac{1}{jw\tau + 1}$$

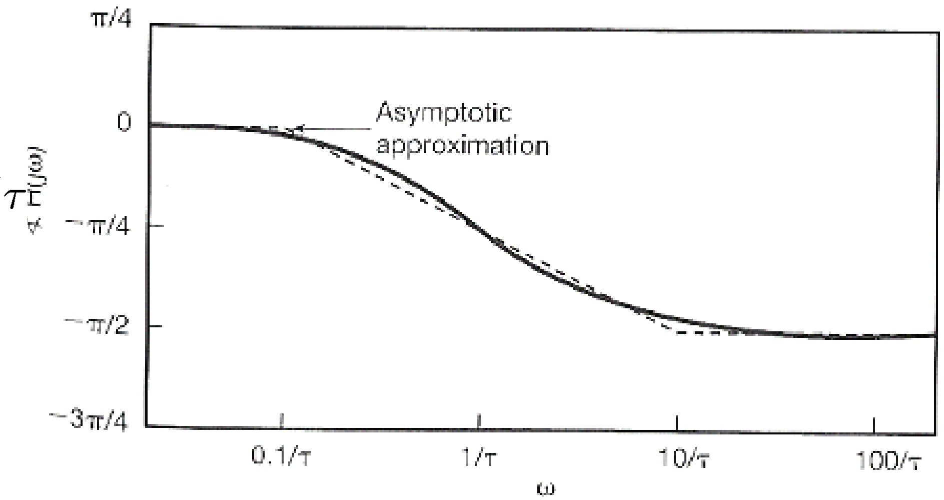
$$20\log_{10}|H(jw)| =$$

$$\begin{cases} 0 & w << 1/\tau \\ -10\log_{10}(2) \approx -3dB & w = 1/\tau \\ -20\log_{10}(w\tau) & w >> 1/\tau \\ \quad = -20\log_{10}(w) - 20\log_{10}(\tau) & \end{cases}$$



$$\measuredangle H(jw) =$$

$$\begin{cases} 0 & w \leq 0.1/\tau \\ -(\pi/4)[\log_{10}(w\tau) + 1] & 0.1/\tau \leq w \leq 10/\tau \\ \quad = -(\pi/4)\left[\log_{10}(w) + \log_{10}(\tau) + 1\right] & \\ -\pi/4 & w = 1/\tau \\ -\pi/2 & w \geq 10/\tau \end{cases}$$



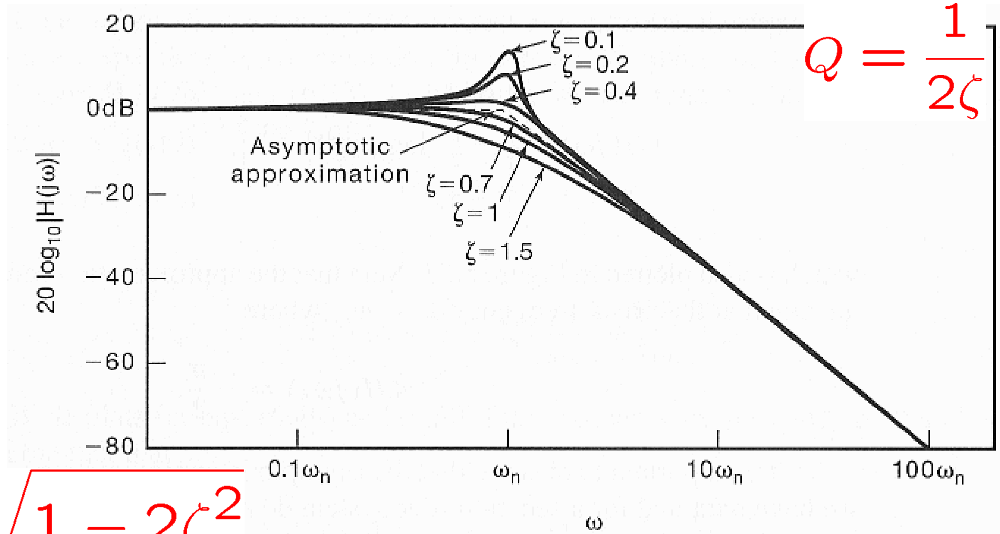
Your Jobs by Hands and Pens

- Second-Order CT Systems:

$$H(jw) = \frac{1}{(j\frac{w}{w_n})^2 + 2\zeta(j\frac{w}{w_n}) + 1}$$

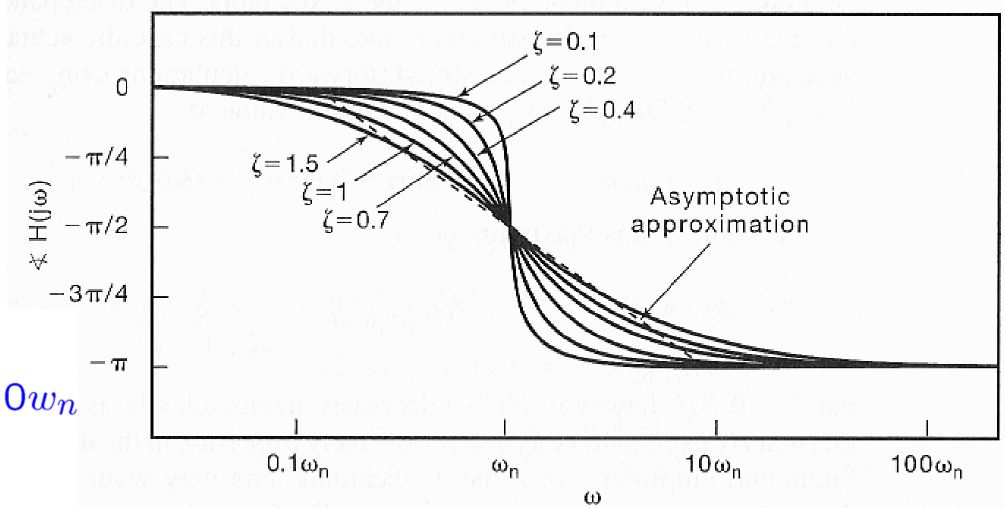
$$Q = \frac{1}{2\zeta}$$

$$20\log_{10}|H(jw)| = \begin{cases} 0 & w << w_n \\ -20\log_{10}(2\zeta) & w = w_n \\ -40\log_{10}(w) + 40\log_{10}(w_n) & w >> w_n \end{cases}$$



- For $\zeta < \frac{\sqrt{2}}{2}$ $\quad w_{\max} = w_n\sqrt{1 - 2\zeta^2}$

$$\measuredangle H(jw) = \begin{cases} 0 & w \leq 0.1w_n \\ -(\pi/2)[\log_{10}(w/w_n) + 1] & 0.1w_n \leq w \leq 10w_n \\ -\pi/2 & w = w_n \\ -\pi & w \geq 10w_n \end{cases}$$



- Example 6.3: Bode Plot for Real Poles and Zeros

$$K\ G(s) = \frac{2000\ (s+0.5)}{s\ (s+10)\ (s+50)}$$

(1) Break points

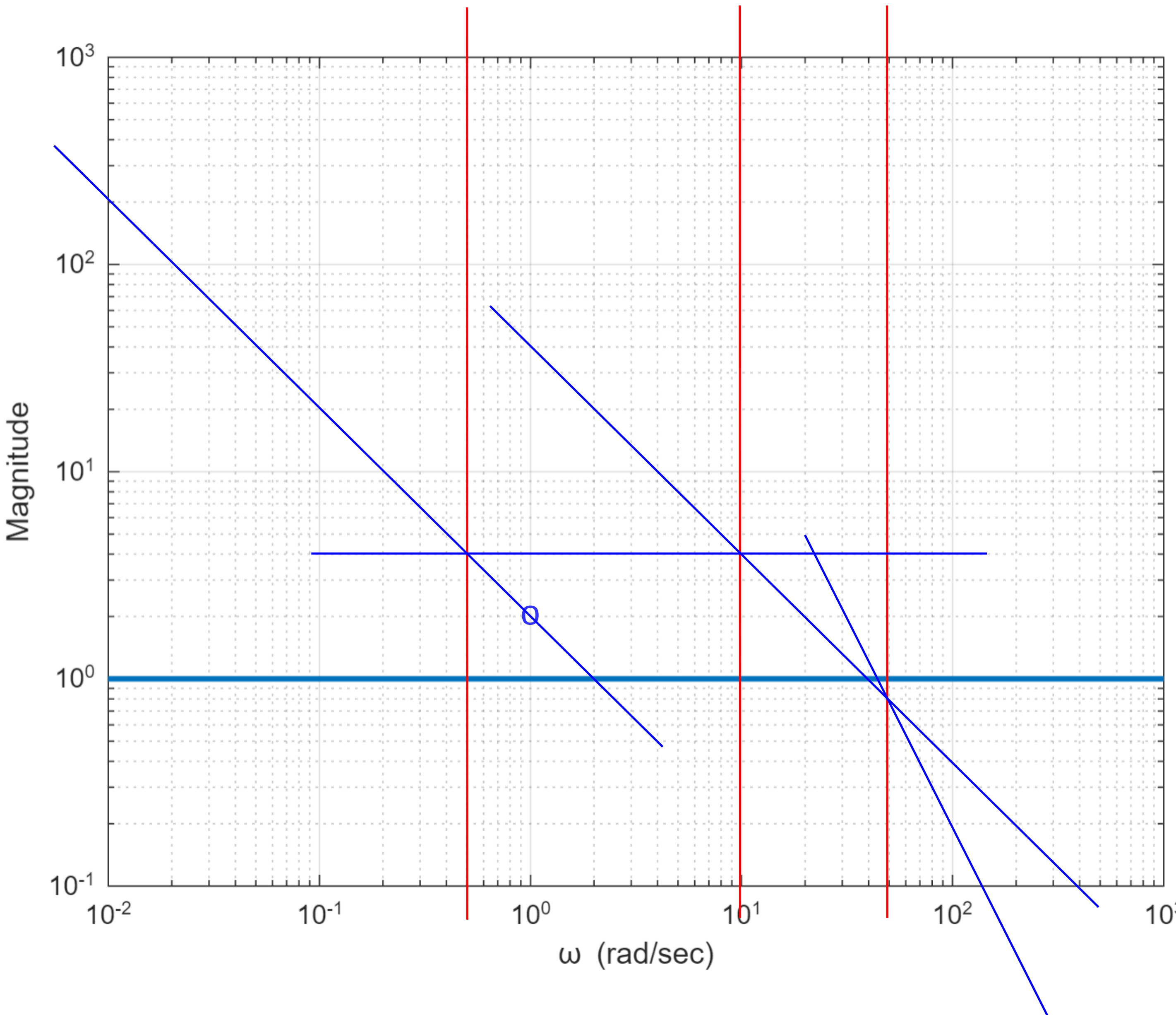
$$K\ G(jw) = \frac{2\ [\ \frac{jw}{0.5} + 1\ ]}{(jw)\ [\ \frac{jw}{10} + 1\ ]\ [\ \frac{jw}{50} + 1\ ]}$$

- Break points: 0.5, 10, 50

(2) Asymptotes

- Low-Frequency Asymptote: $K\ G(jw) = \frac{2}{(jw)}$ for $w < 0.1$
- ω << 0.5: slope = -1 (or -20 db/decade)
- 0.5 < ω < 10: slope = 0 (or 0 db/decade)
- 10 < ω < 50: slope = -1 (or -20 db/decade)
- 50 < ω: slope = -2 (or -40 db/decade)

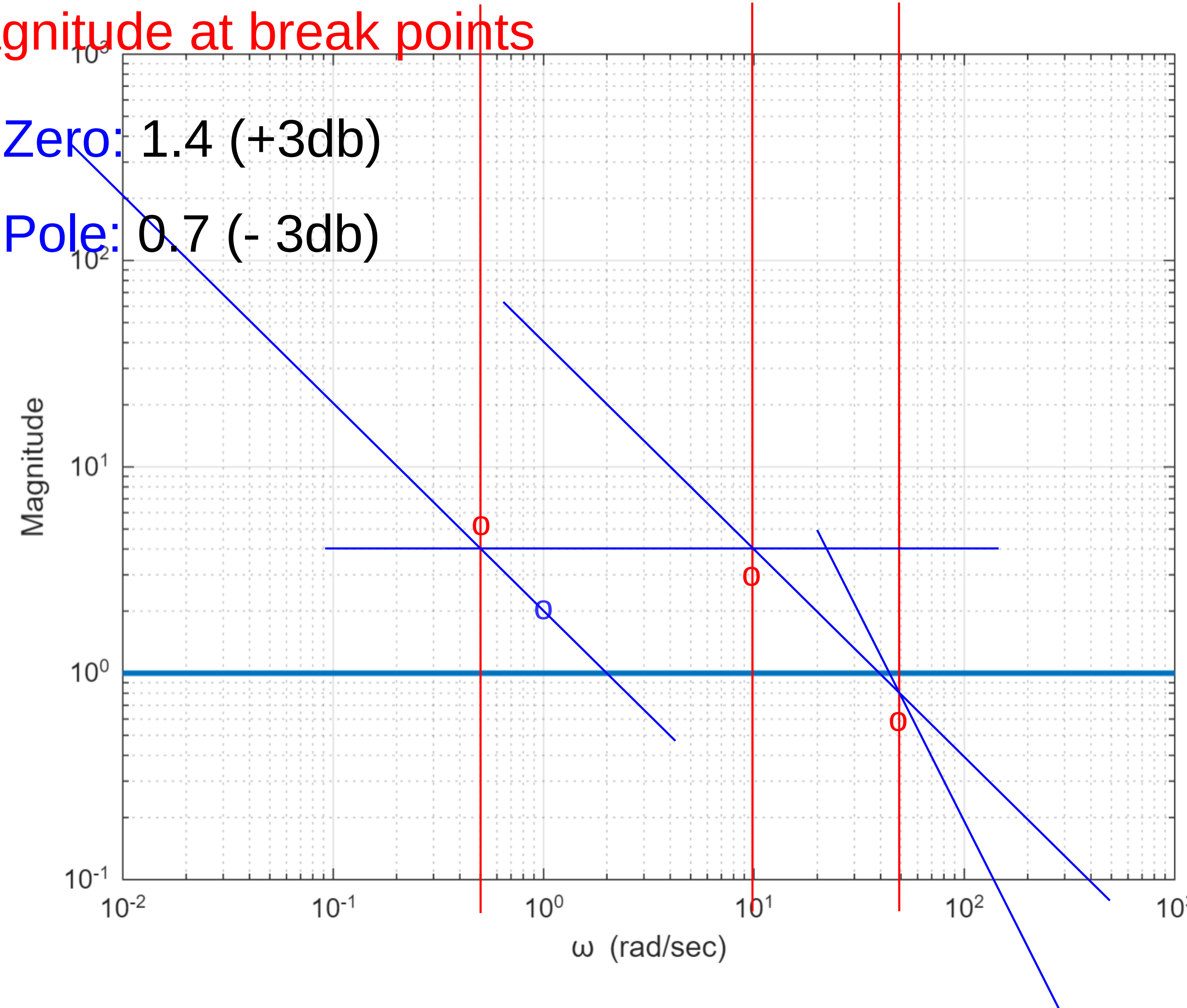
- Example 6.3: Bode Plot for Real Poles and Zeros



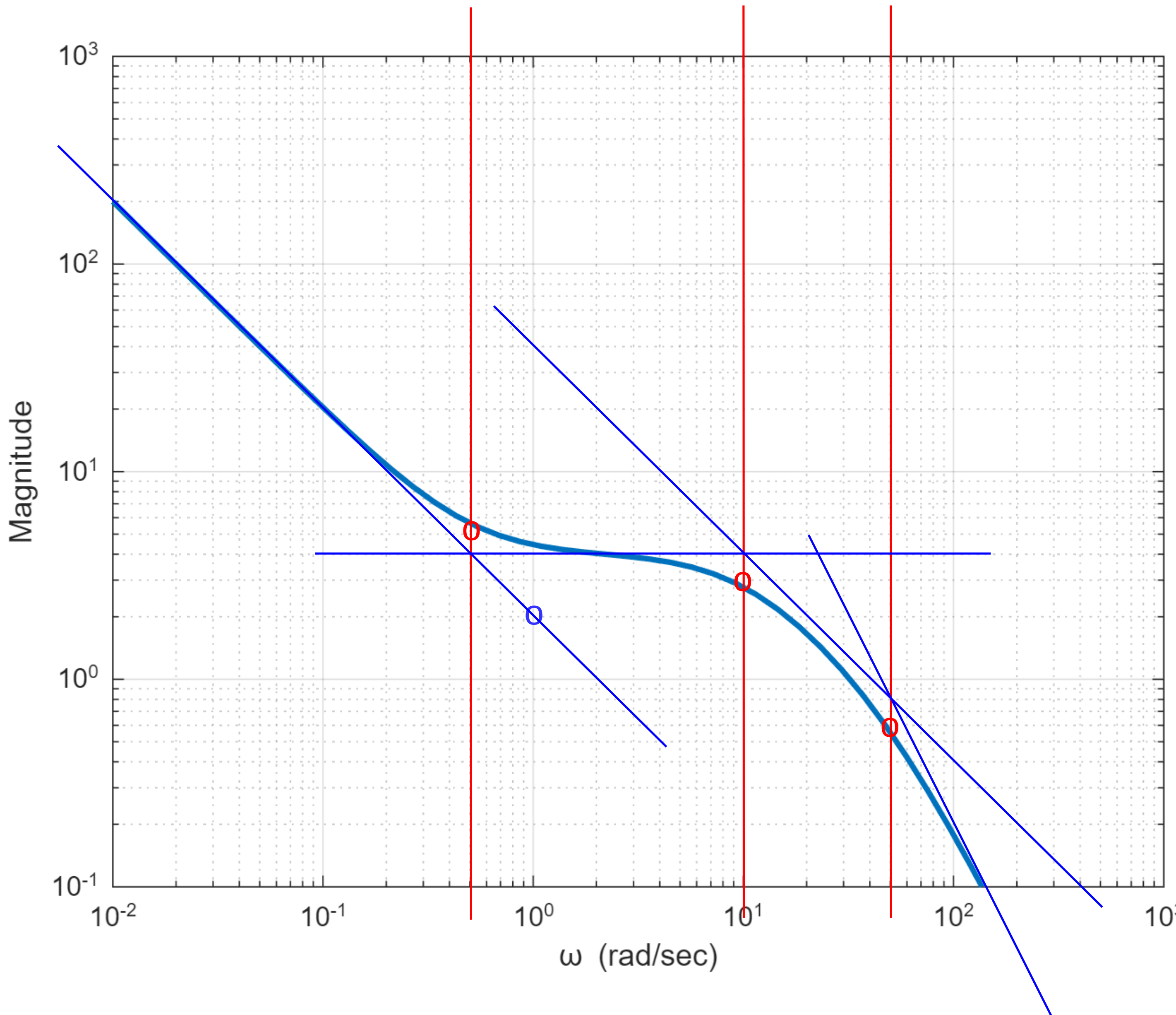
- Example 6.3: Bode Plot for Real Poles and Zeros

(3) Magnitude at break points

- By Zero: 1.4 (+3db)
- By Pole: 0.7 (- 3db)



- Example 6.3: Bode Plot for Real Poles and Zeros



- Example 6.3: Bode Plot for Real Poles and Zeros

$$K\ G(s) = \frac{2000\ (s+0.5)}{s\ (s+10)\ (s+50)}$$

(1) Break points

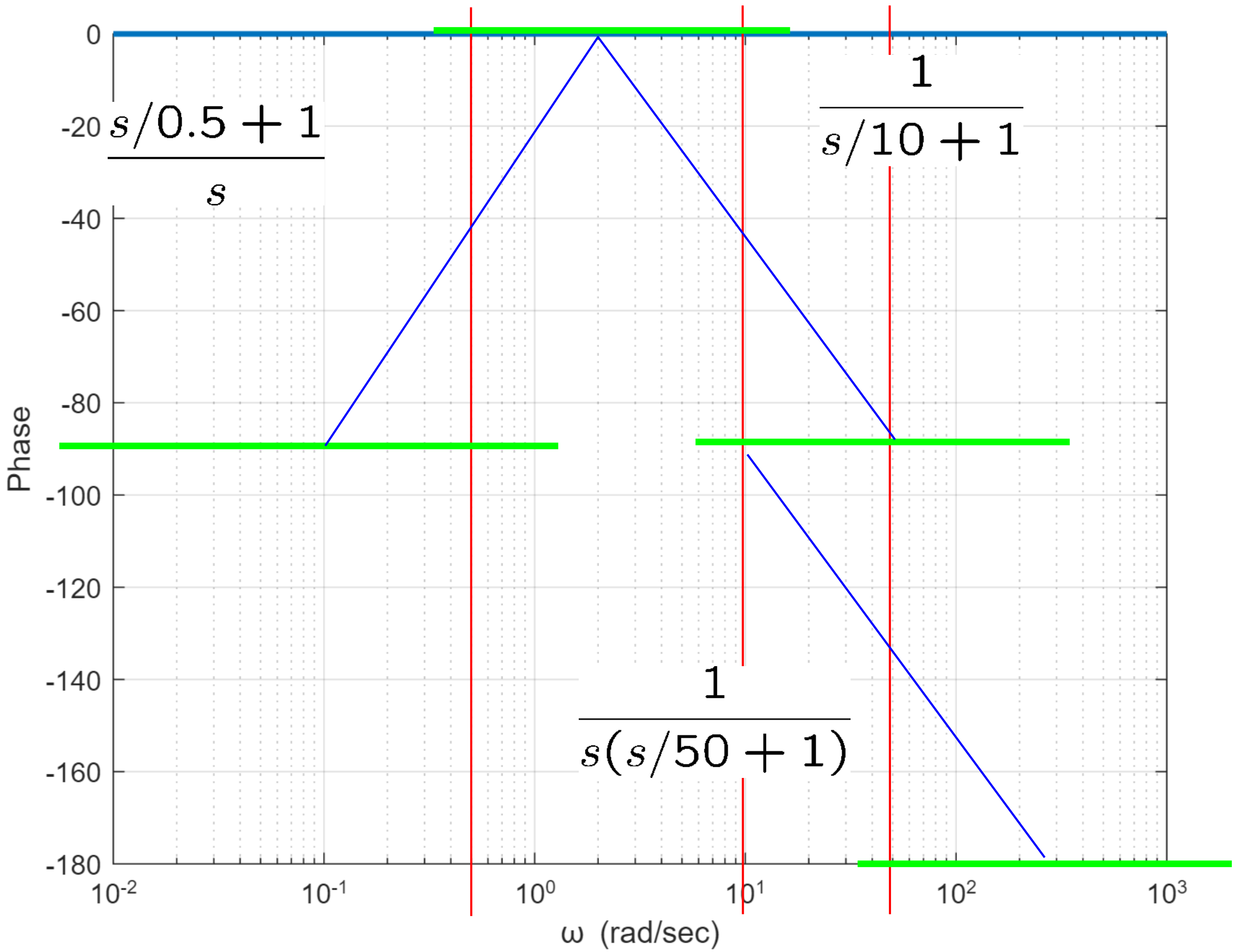
$$K\ G(jw) = \frac{2\ [\ \frac{jw}{0.5}+1\ ]}{(jw)\ [\ \frac{jw}{10}+1\ ]\ [\ \frac{jw}{50}+1\ ]}$$

- Break points: 0.5, 10, 50

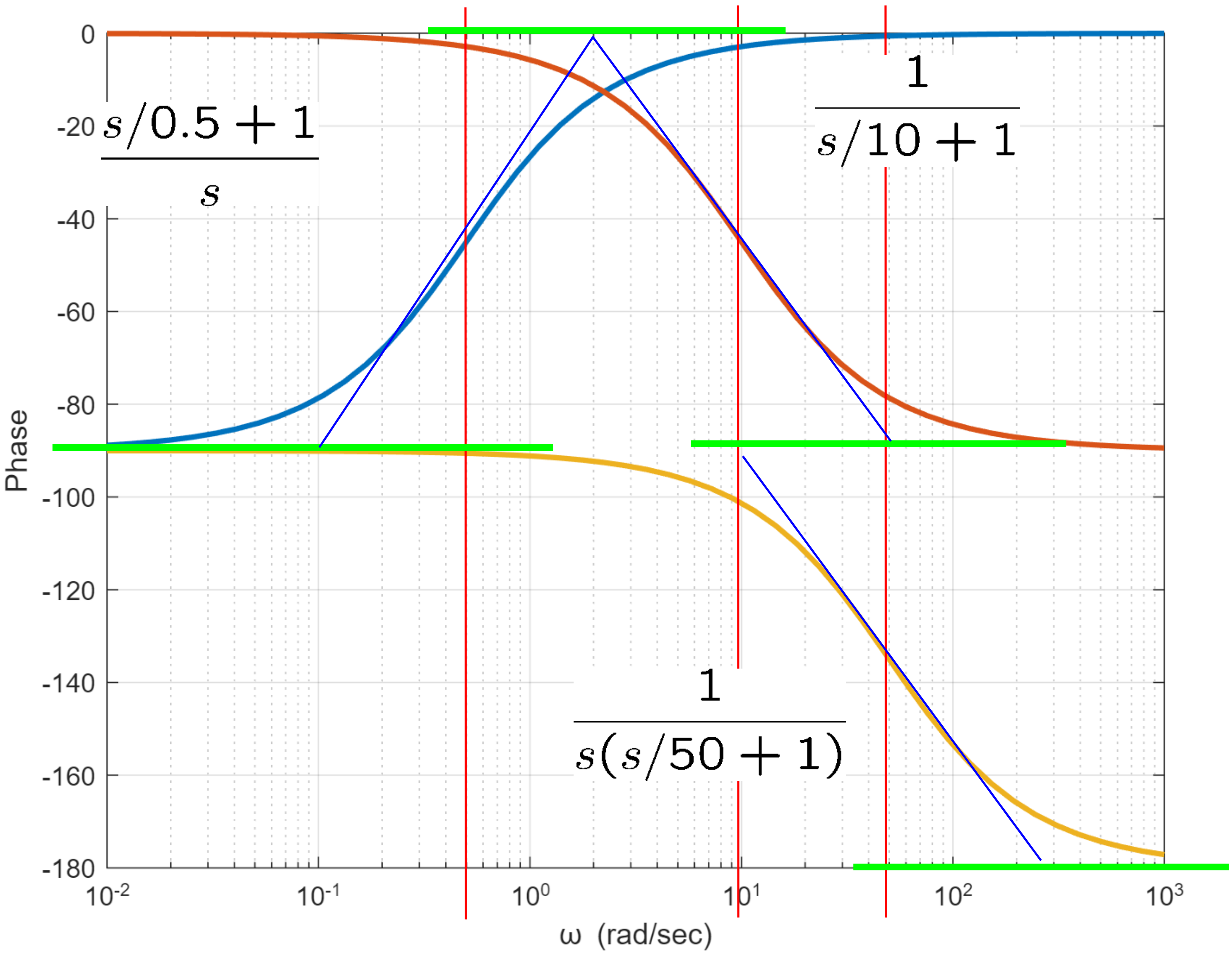
(4) Phase

- Low-Frequency Asymptote: $K\ G(jw) = \frac{2}{(jw)}$ for $w < 0.1$
- $\omega << 0.5$: phase = -90^o
- $0.5 < \omega < 10$: phase = 0^o
- $10 < \omega < 50$: phase = -90^o
- $50 < \omega$: phase = -180^o

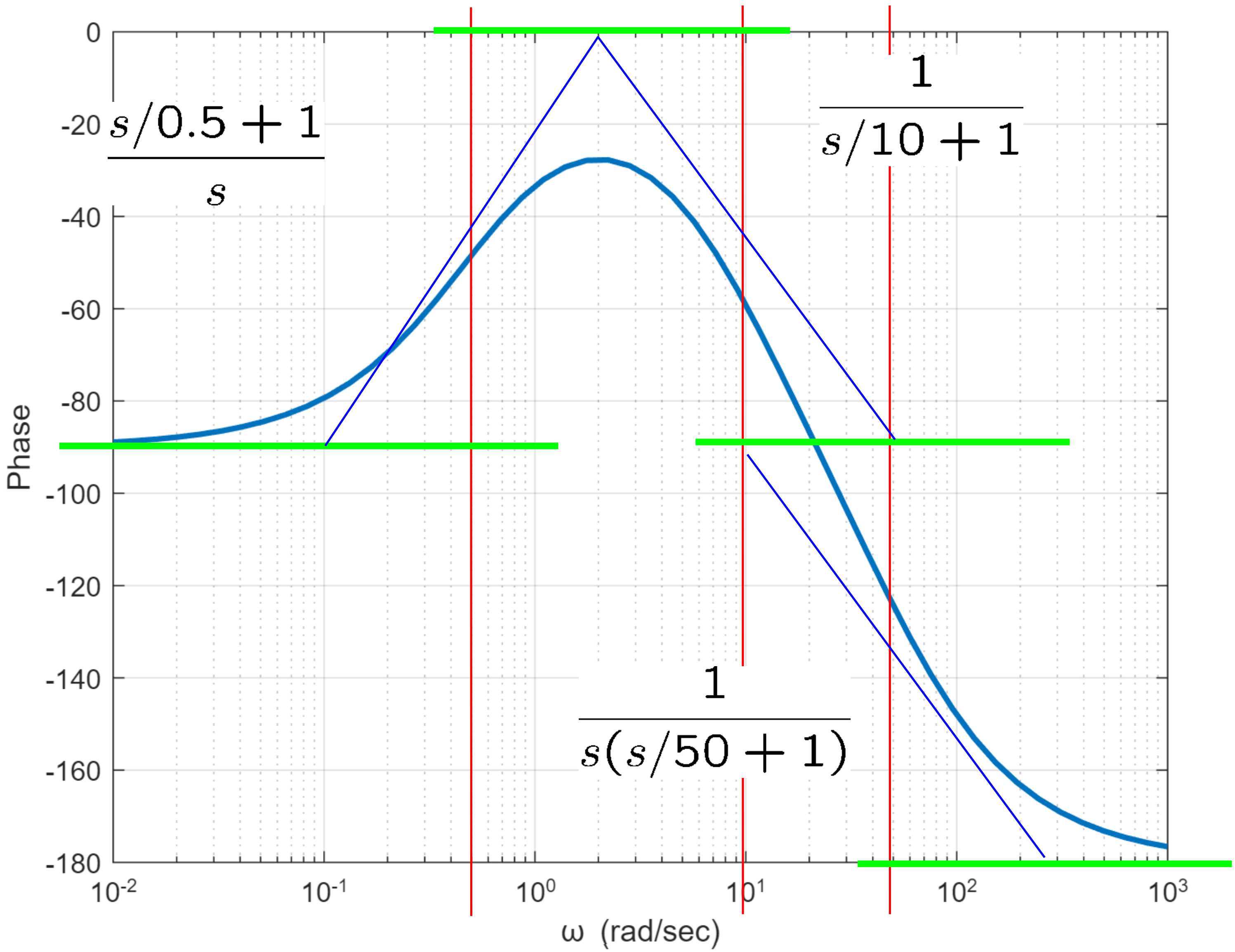
- Example 6.3: Bode Plot for Real Poles and Zeros



- Example 6.3: Bode Plot for Real Poles and Zeros



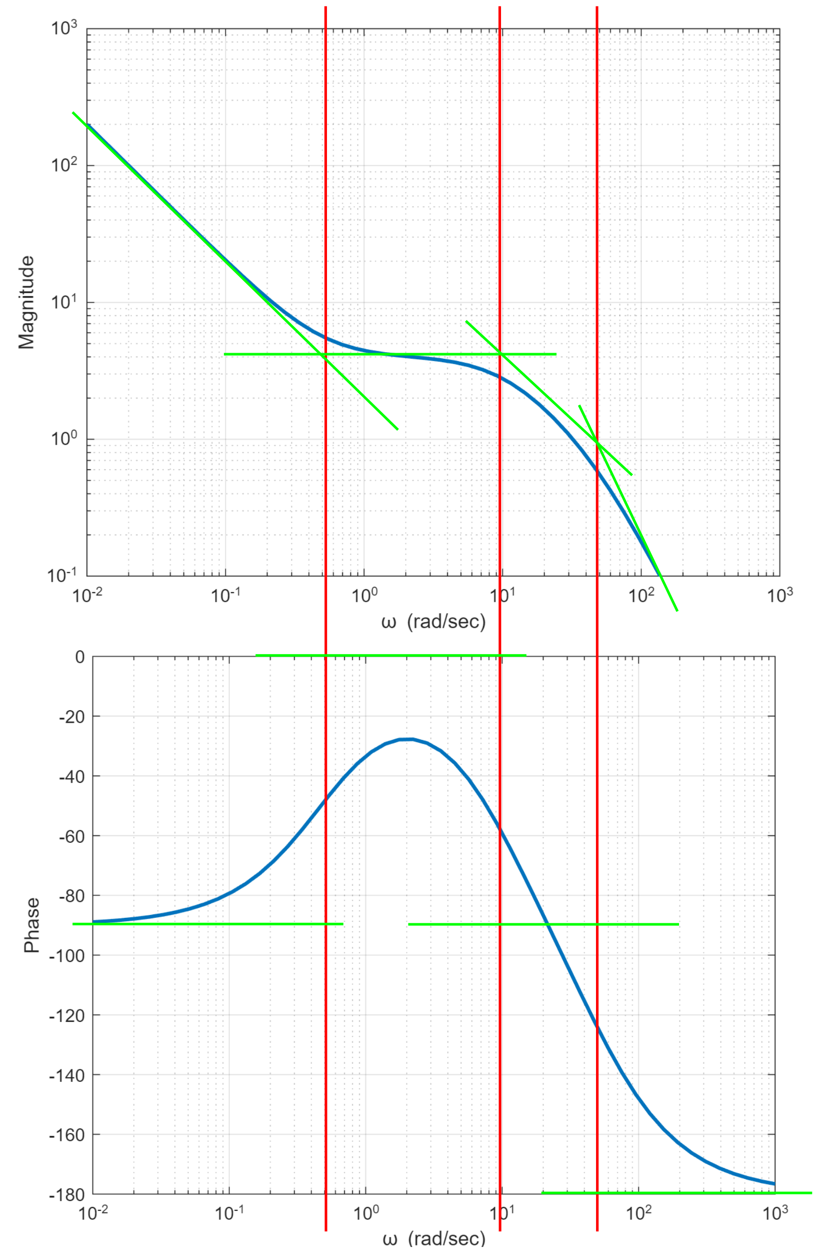
- Example 6.3: Bode Plot for Real Poles and Zeros



- Example 6.3: Bode Plot for Real Poles and Zeros

$$\frac{2000\,(s+0.5)}{s\,(s+10)\,(s+50)}$$

$$\frac{2\,[\,\frac{jw}{0.5}+1\,]}{(jw)\,[\,\frac{jw}{10}+1\,]\,[\,\frac{jw}{50}+1\,]}$$

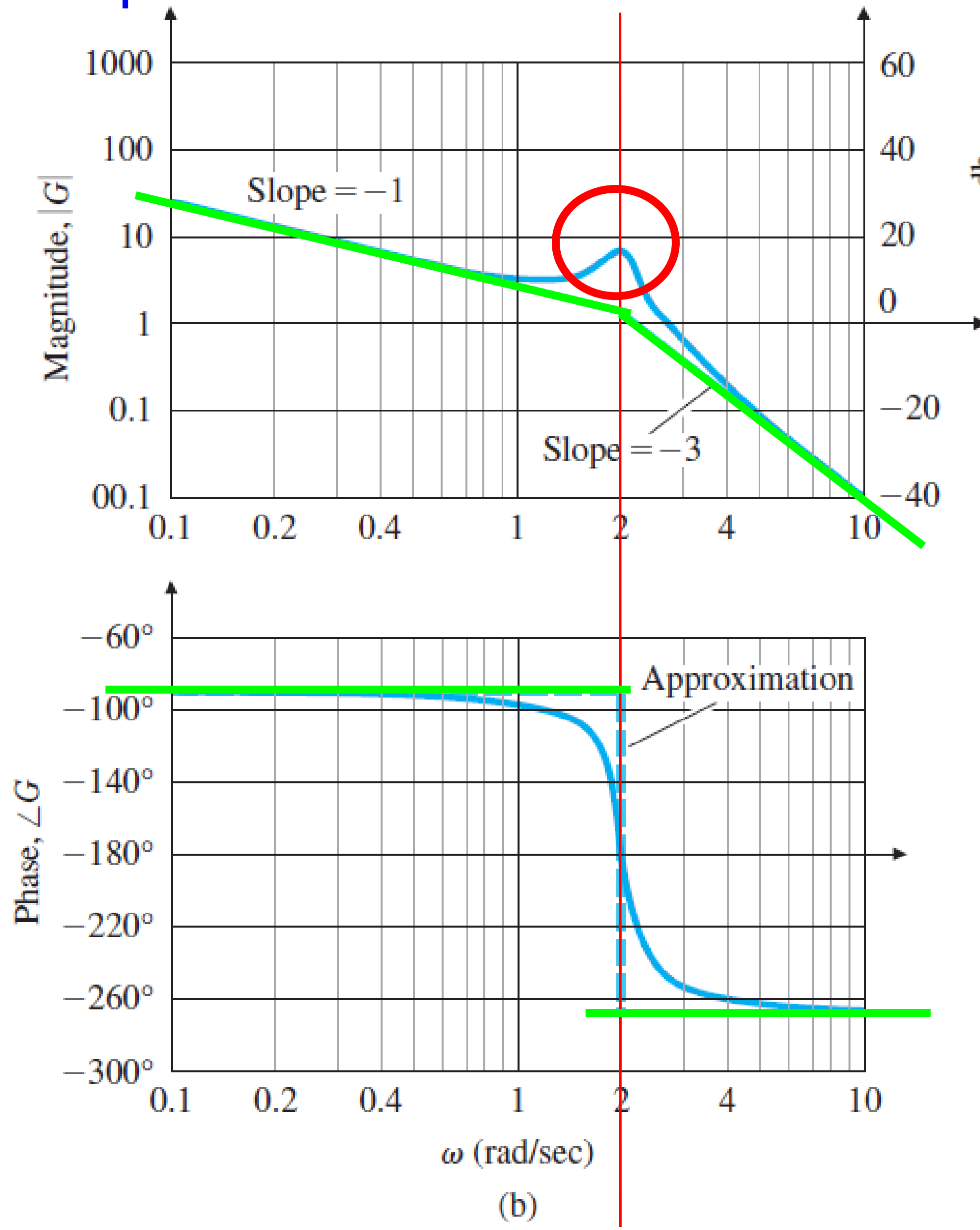


- Break points: 0.5, 10, 50

- Example 6.4: Bode Plot for Complex Poles

$$K\,G(s) = \frac{10}{s\,[s^2 + 0.4s + 4]}$$

$$= \frac{10}{4}\,\frac{1}{s\,[\,\frac{s^2}{4} + 2(0.1)\frac{s}{2} + 1\,]}$$

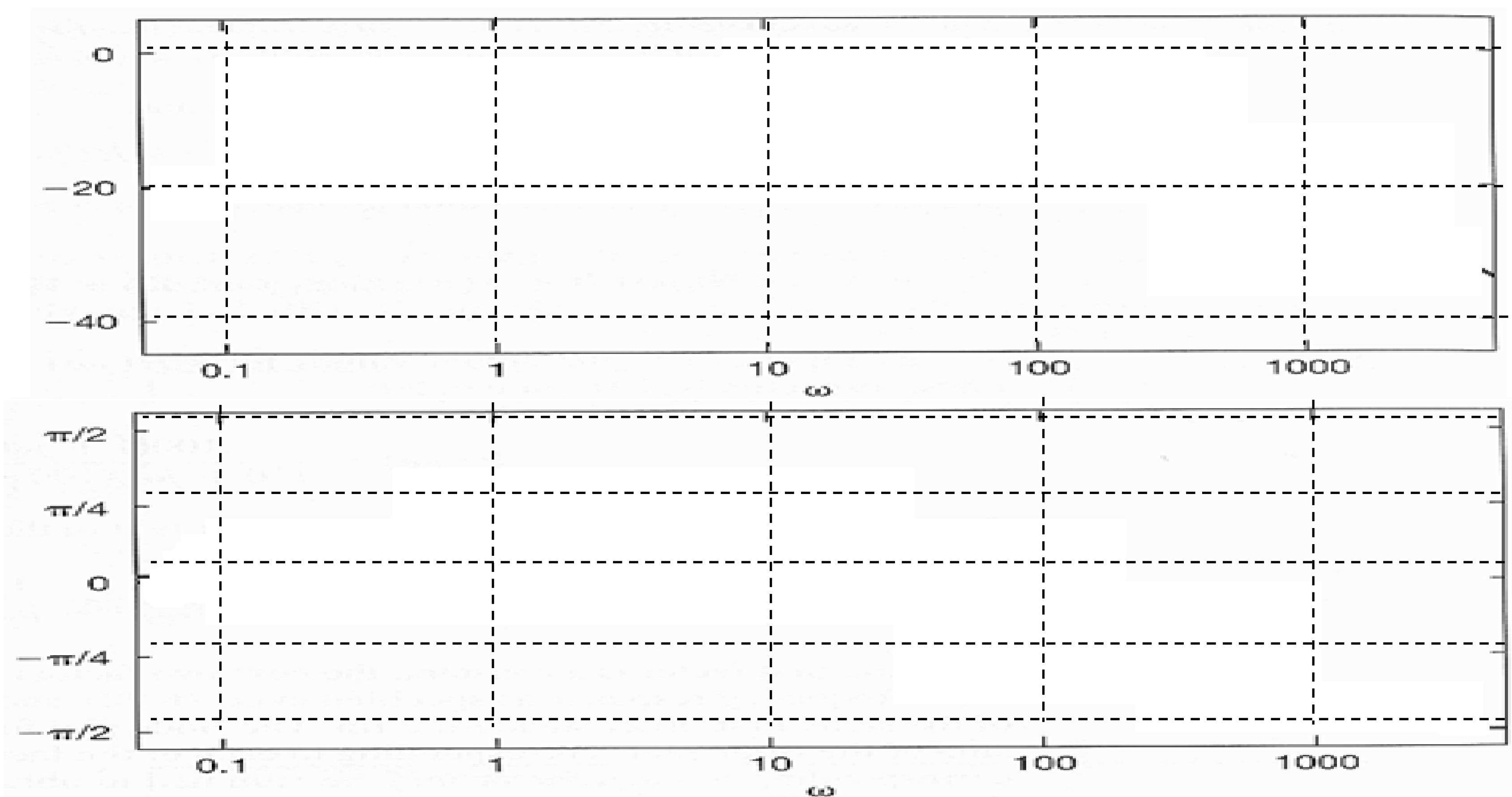


- Break points: 2

- Example 6.5:

$$H(jw) = \frac{100(1+jw)}{(10+jw)(100+jw)}$$

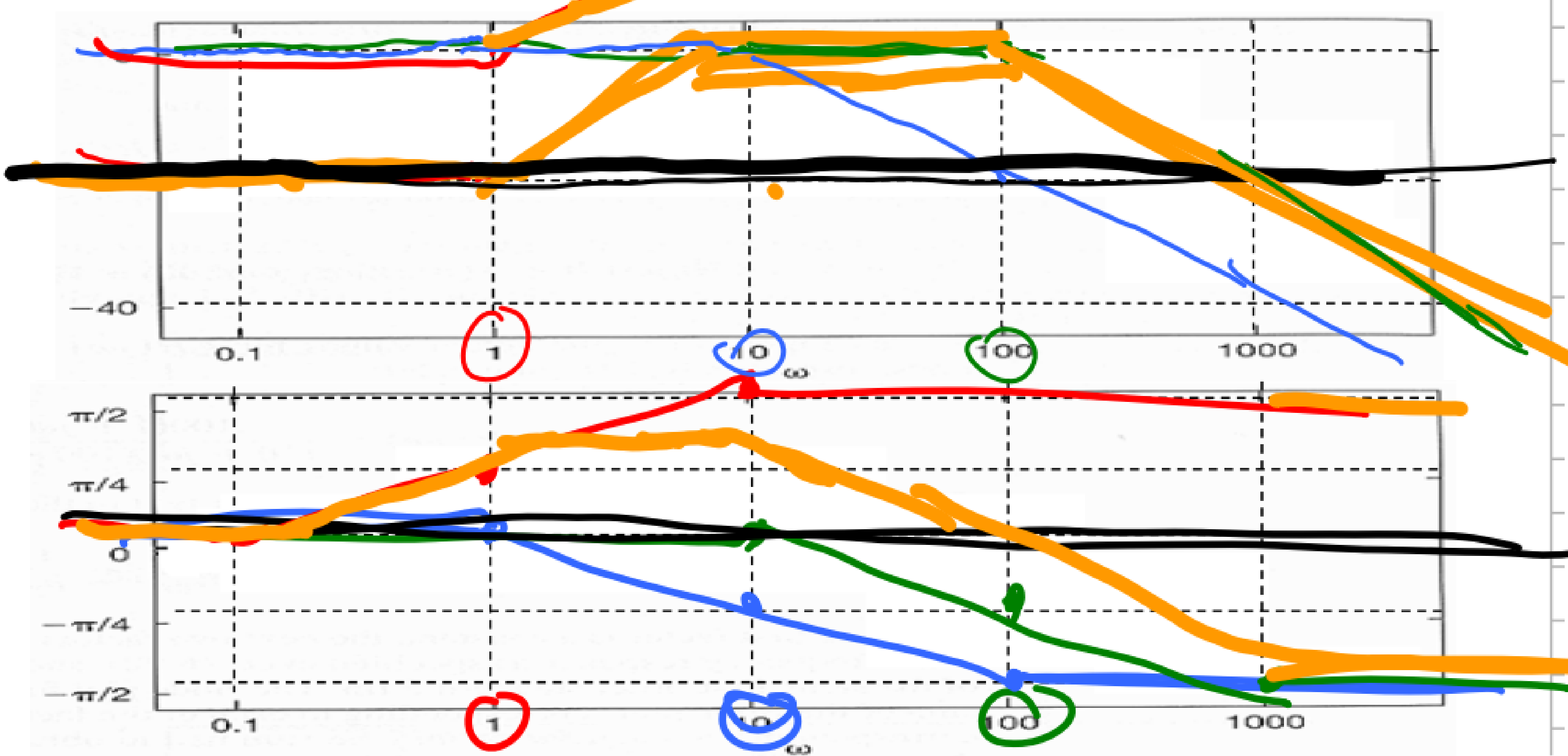
$$= \left(\frac{1}{10}\right)(1+jw)\left(\frac{1}{1+j\frac{w}{10}}\right)\left(\frac{1}{1+j\frac{w}{100}}\right)$$



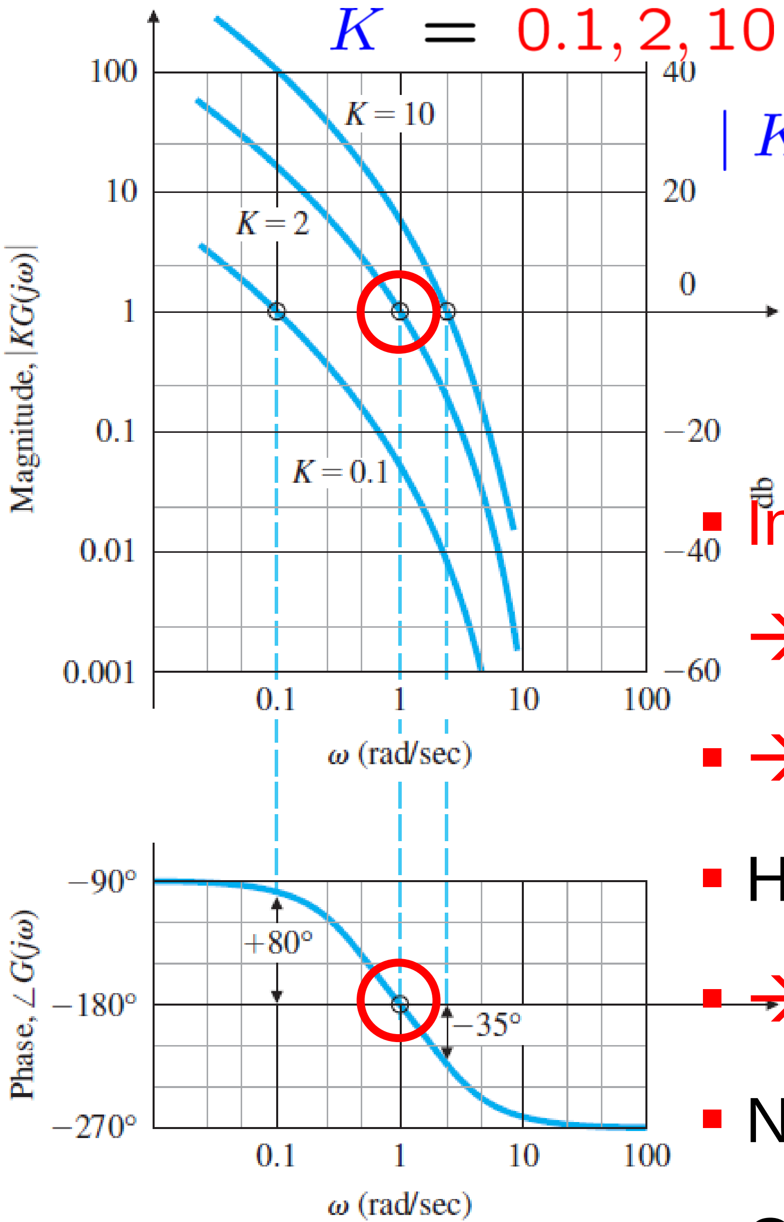
- Example 6.5:

$$H(jw) = \frac{100(1+jw)}{(10+jw)(100+jw)}$$

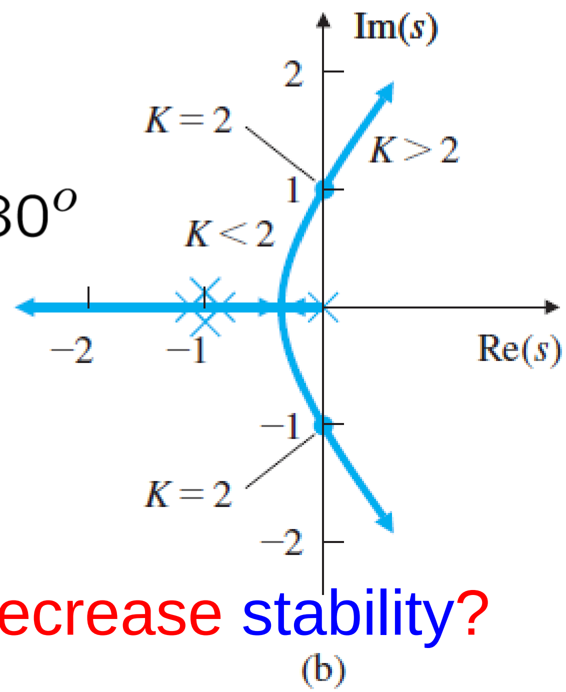
$$= \left(\frac{1}{10}\right)(1+jw)\left(\frac{1}{1+j\frac{w}{10}}\right)\left(\frac{1}{1+j\frac{w}{100}}\right)$$



$K = 0.1, 2, 10$

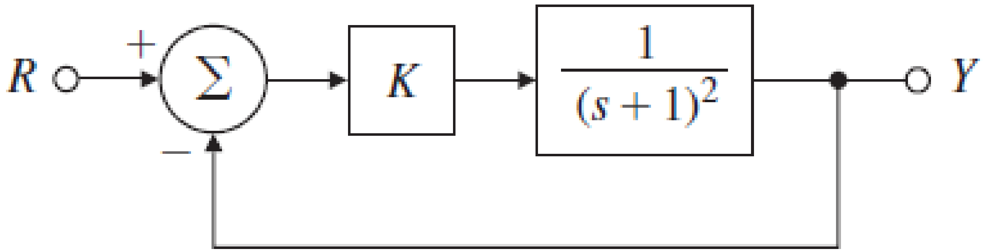


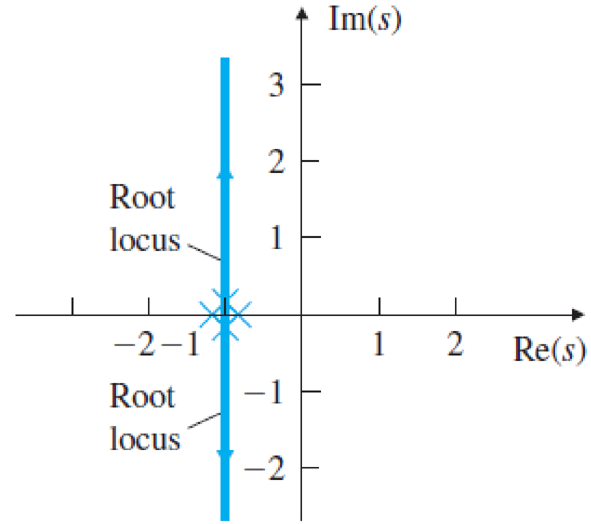
$$|K\,G(jw)| = 1$$
$$\angle G(jw) = 180^o$$

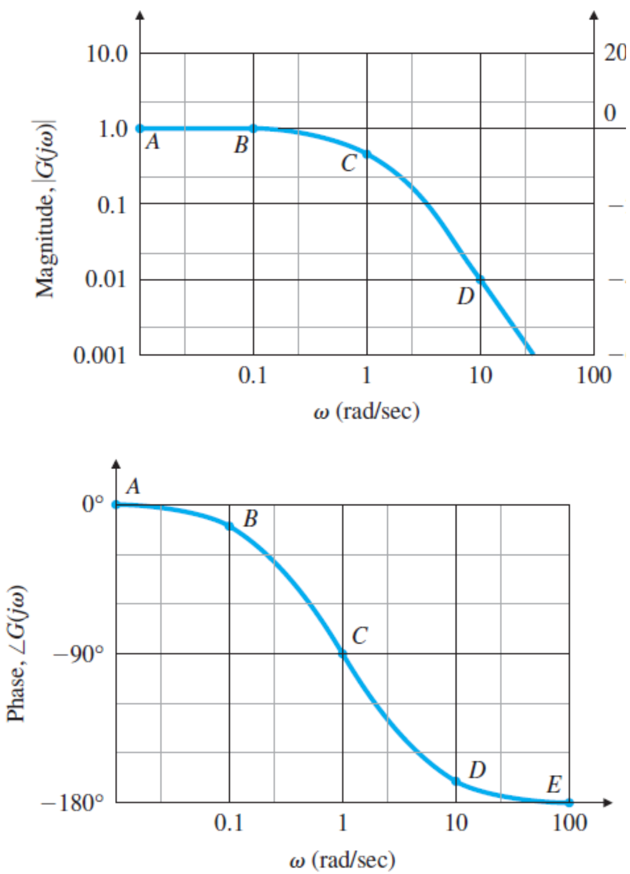


- Increasing gain
  → increase OR decrease stability?
- → $|KG(jw)| < 1$!!!
- However, for some systems,
- → $|KG(jw)| > 1$!!!
- Need to check their root locus
- OR, by Nyquist Stability Criterion

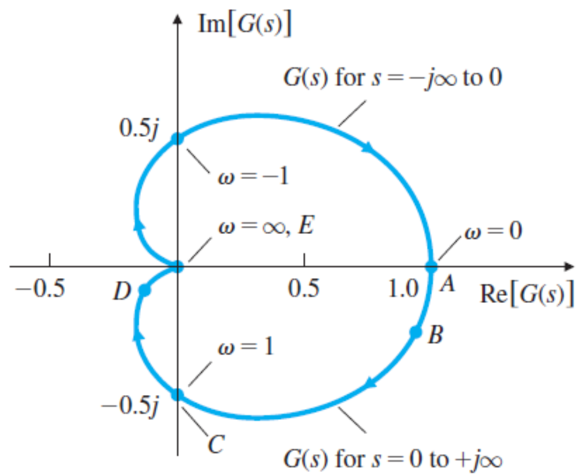
Example 6.8: Nyquist Plot for a Second-Order System





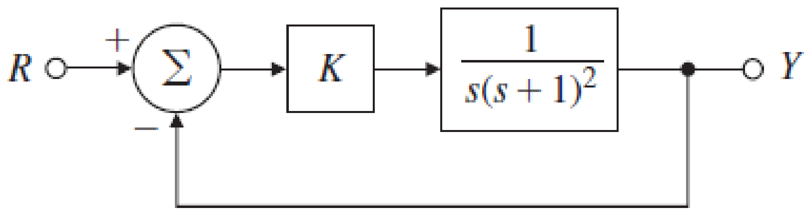


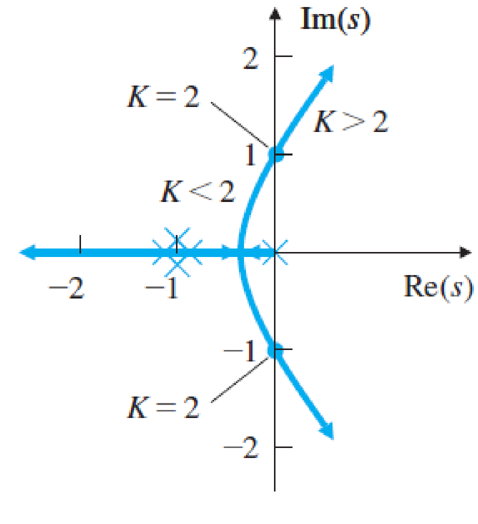
(b)



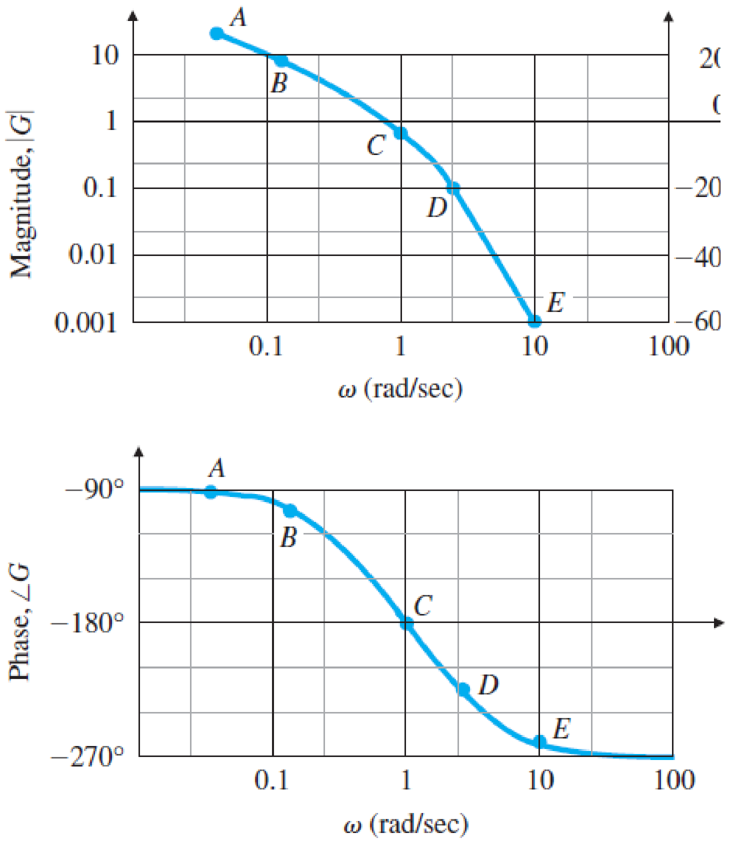
- N = 0: not encircle -1
- P = 0: no poles of G(s) in RHP
- Z = N + P → Z = 0,
  no unstable roots for K = 1
- K > 0 also holds

Example 6.9: Nyquist Plot for a Third-Order System

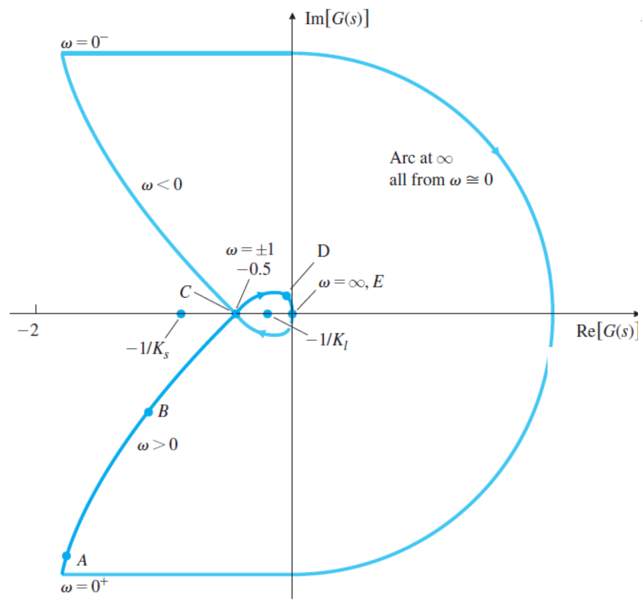


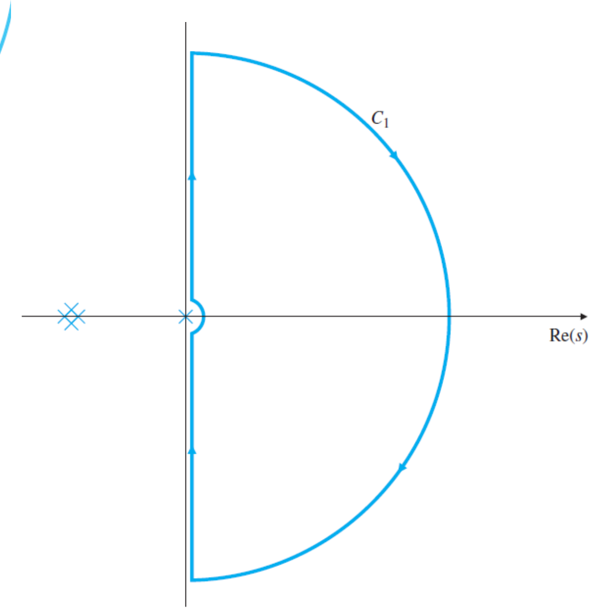


(b)

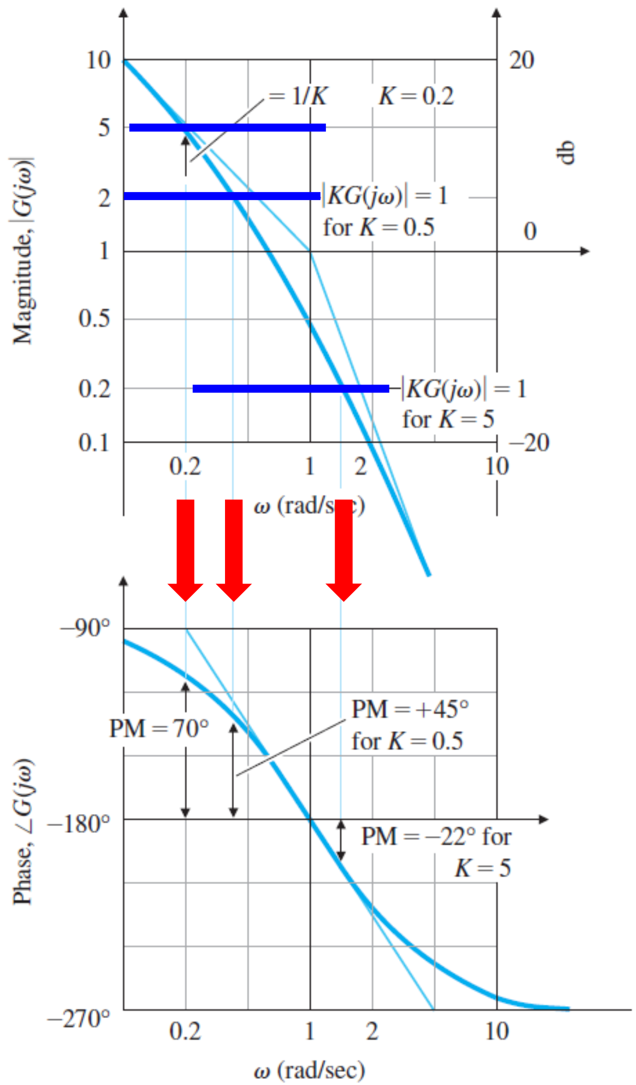


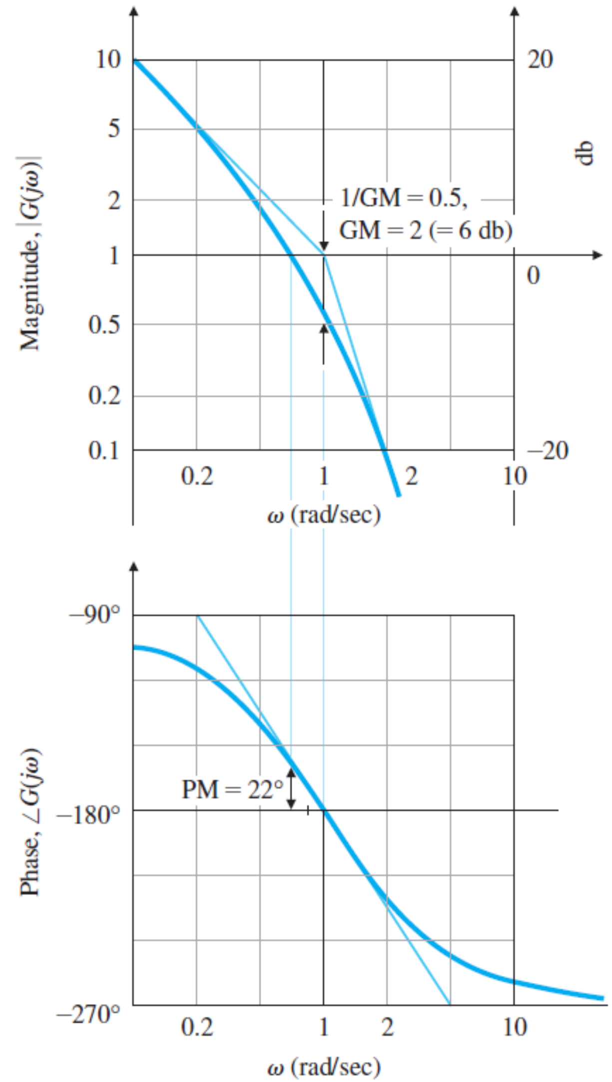
(b)





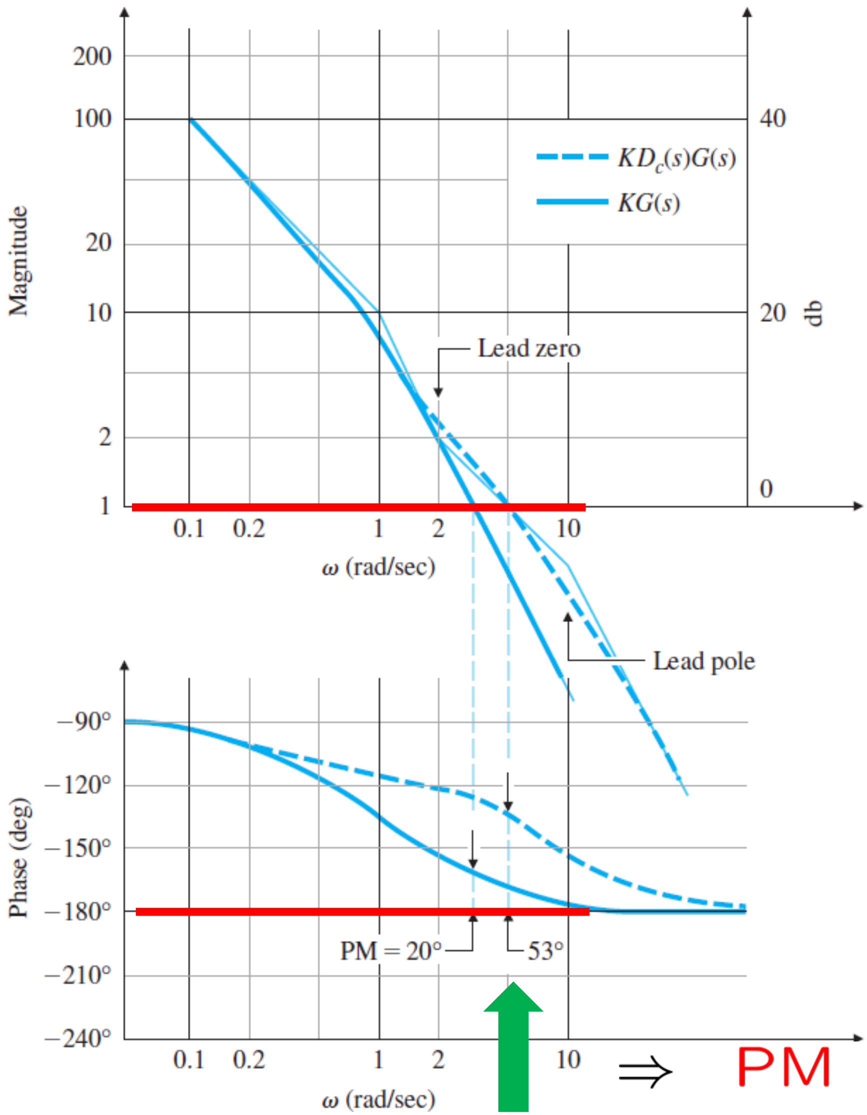
- PM vs K
- K = 5
- | KG(jw) | = 1
- PM = -22^o
- K = 0.5
- | KG(jw) | = 1
- PM = $+45^o$
- K = 0.2
- | KG(jw) | = 1
- PM = $+70^o$





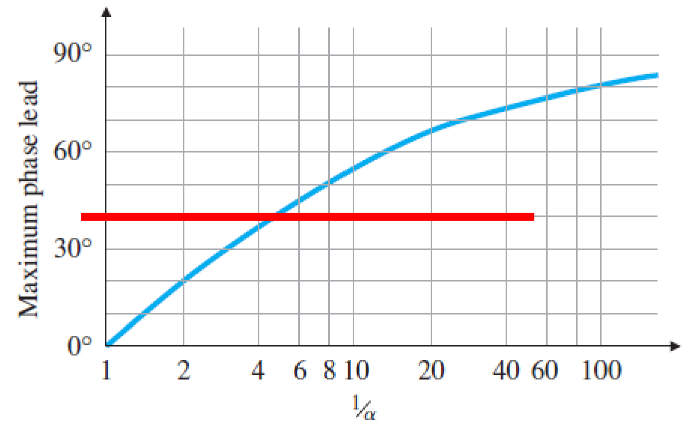
▪ Example 6.15: Lead Compensation for a DC Motor

$K\ G(s)$ $\qquad$ $K\ D_c(s)\ G(s)$



$\Rightarrow$ PM $\geq 25^o$, at $w_c = 3$

$\Rightarrow$ Phase lead $= 40^o$,



$$\Rightarrow \frac{1}{\alpha} = 5$$

a zero at $w = 2$ rad/sec

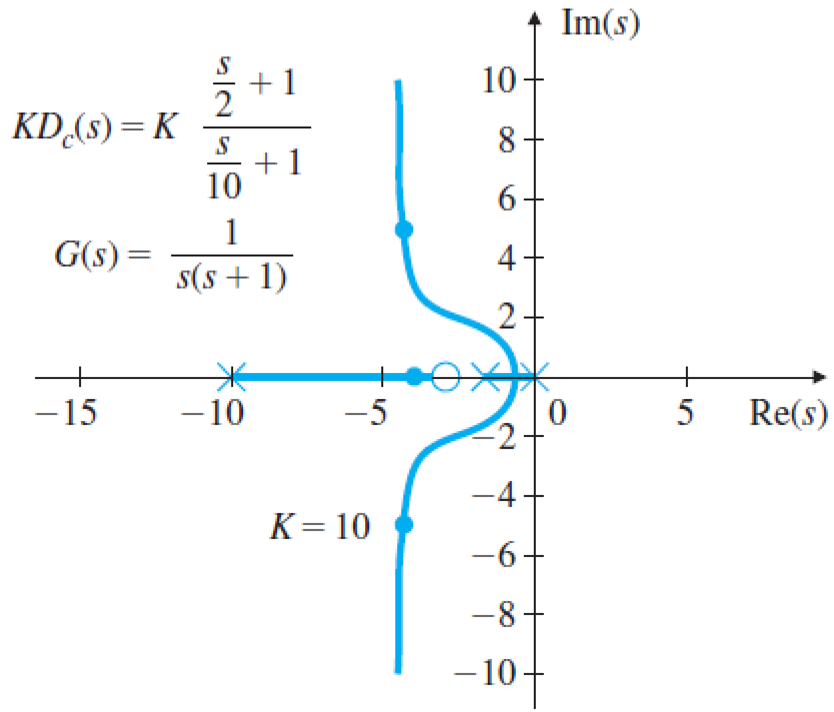
a pole at $w = 10$ rad/sec

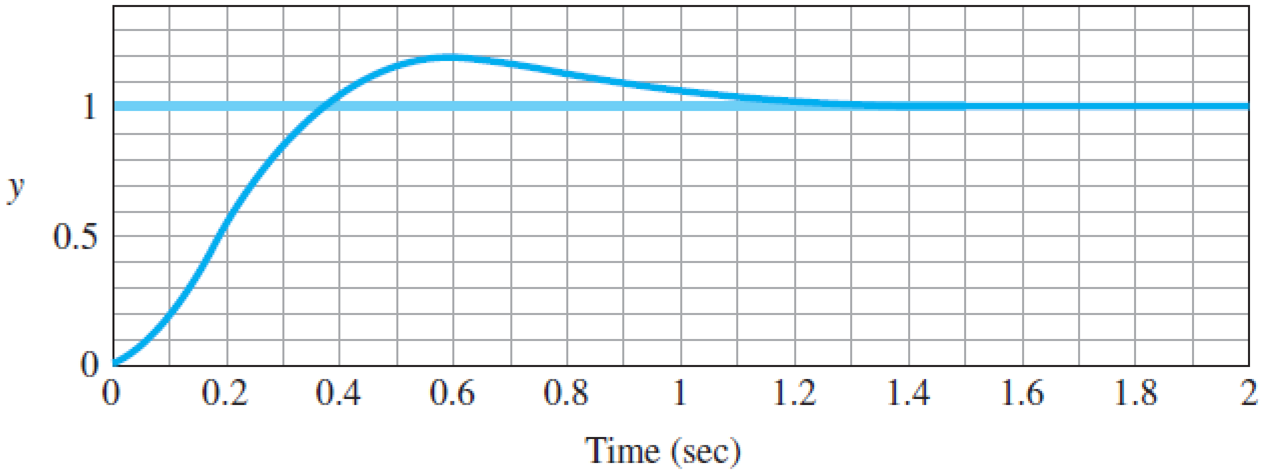
$$K\ D_c(s) = 10\ \frac{\frac{s}{2}+1}{\frac{s}{10}+1}$$

$\Rightarrow$ PM $= 53^o$, at $w_c = 5$

Example 6.15: Lead Compensation for a DC Motor

$$K\ D_c(s) = 10\,\frac{\frac{s}{2}+1}{\frac{s}{10}+1}$$





▪ Example 6.17: Lead-Compensation Design for Type 1 Servomechanism System

$$K\,G(s) = K \frac{10}{s\,(\frac{s}{2.5}+1)\,(\frac{s}{6}+1)}$$

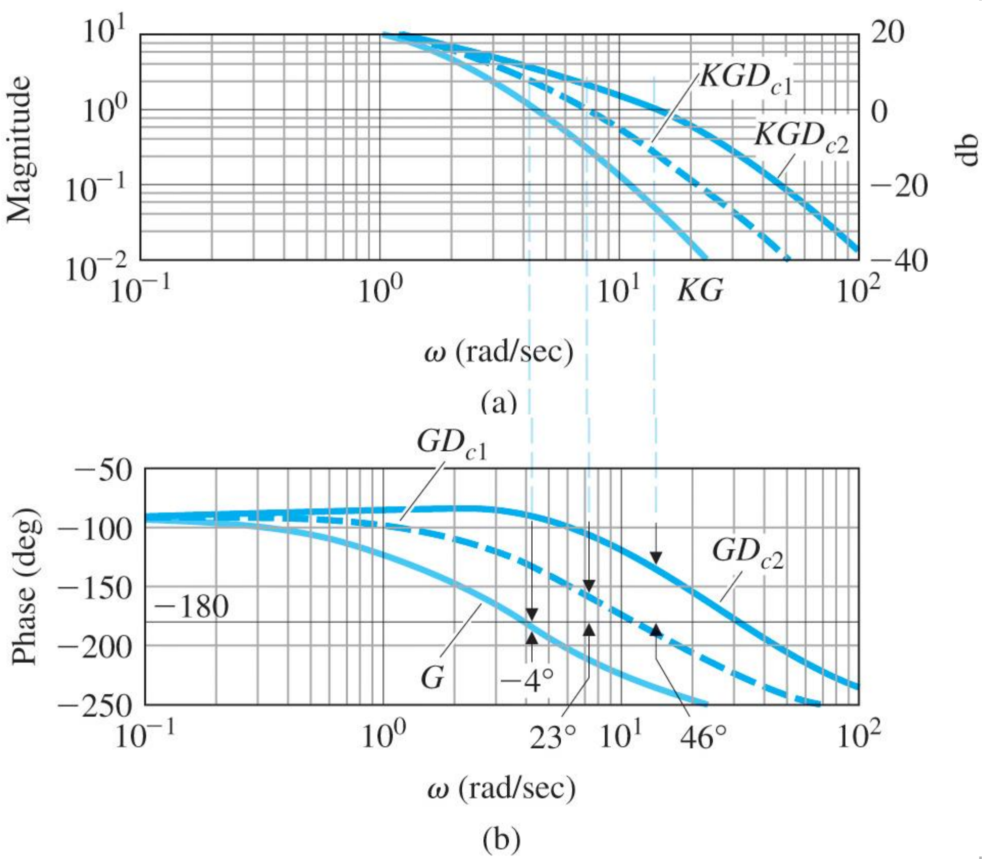
- $K_v = 10$
- $\text{PM} = 45^o$

1. Determine gain K:

$$K_v = \lim_{s \to 0} s\,K\,G(s) = K \times 10 = 10 \Rightarrow K = 1$$

2. Bode plot of KG(s), K = 1

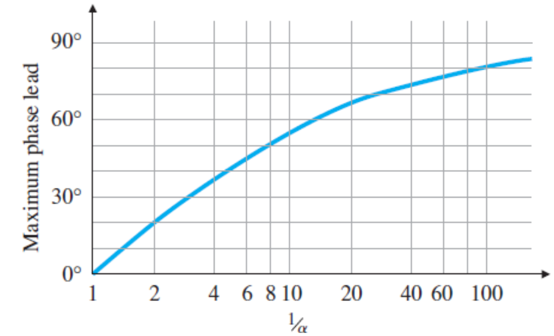
→ PM ~= - 4, Wcp ~= 4



▪ Example 6.17: Lead-Compensation Design for Type 1 Servomechanism System

3. Allow for 5^o of extra margin

→ $45^o + 5^o - (-4^o) = 54^o$



4. Pick α → 1/α = 10

5. Zero & Pole

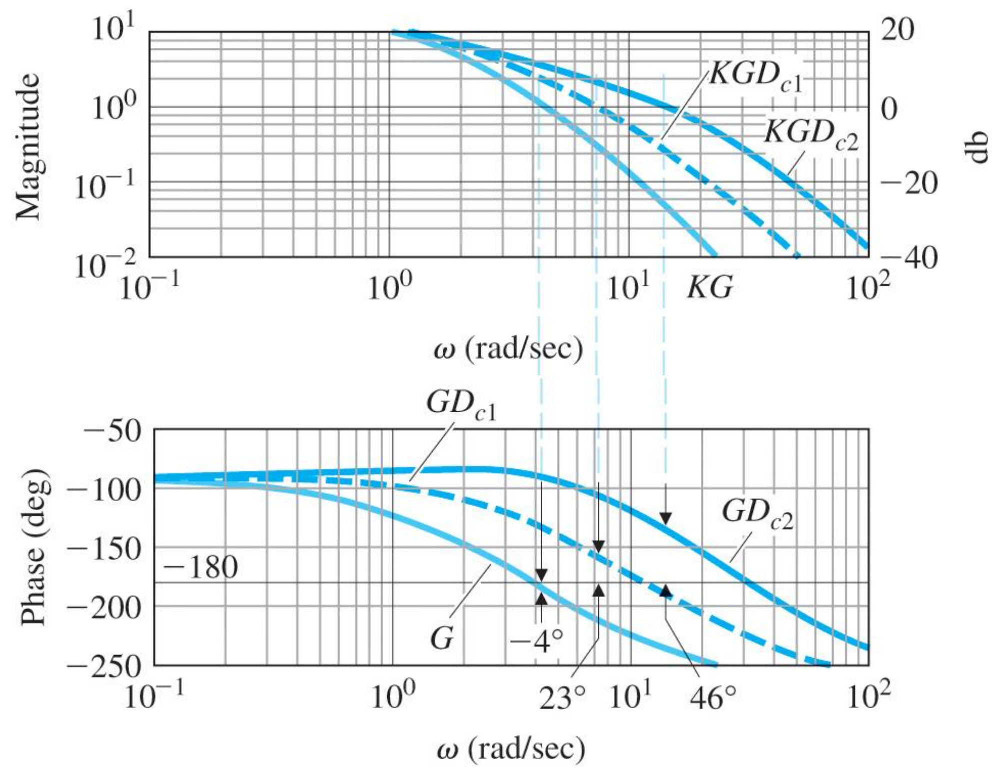
a zero at 2

a pole at 20

$$D_1(s) = \frac{(\frac{s}{2}+1)}{(\frac{s}{20}+1)}$$

$$= \frac{1}{0.1}\left(\frac{s+2}{s+20}\right)$$

→ PM ~= 23, Wcp ~= 7



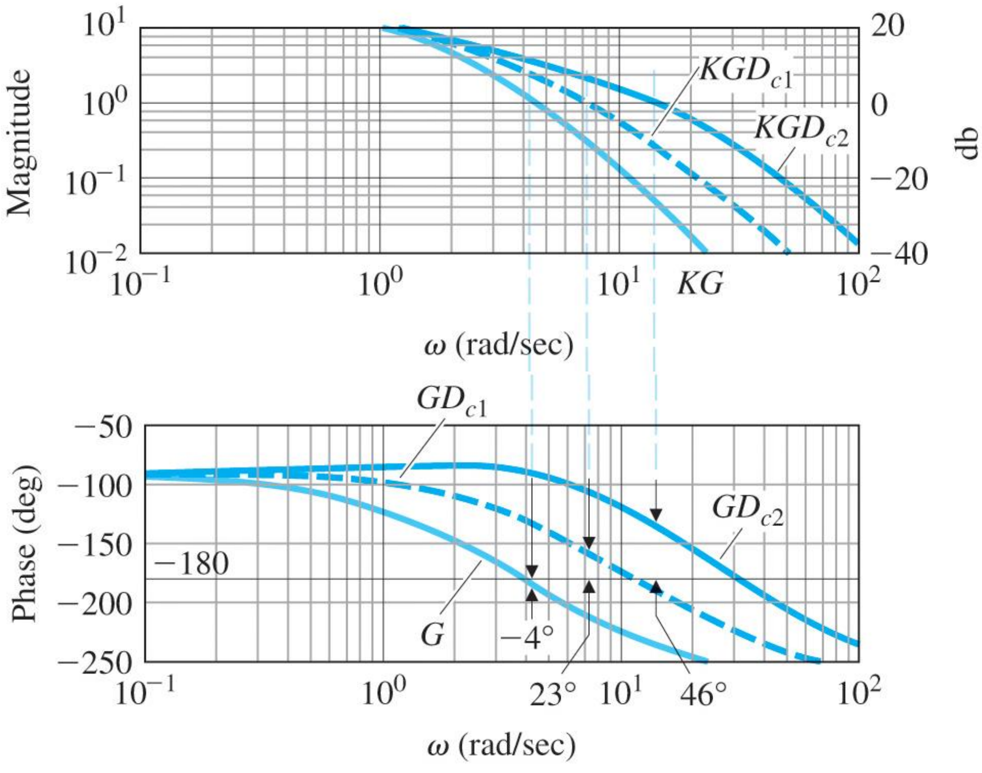
(b)

- Example 6.17: Lead-Compensation Design for Type 1 Servomechanism System

7. A double-lead compensator:

$$D_2(s) = \frac{\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)}{\left(\frac{s}{20}+1\right)\left(\frac{s}{40}+1\right)} = \frac{1}{(0.1)^2}\frac{(s+2)(s+4)}{(s+20)(s+40)}$$

- PM = 46^o

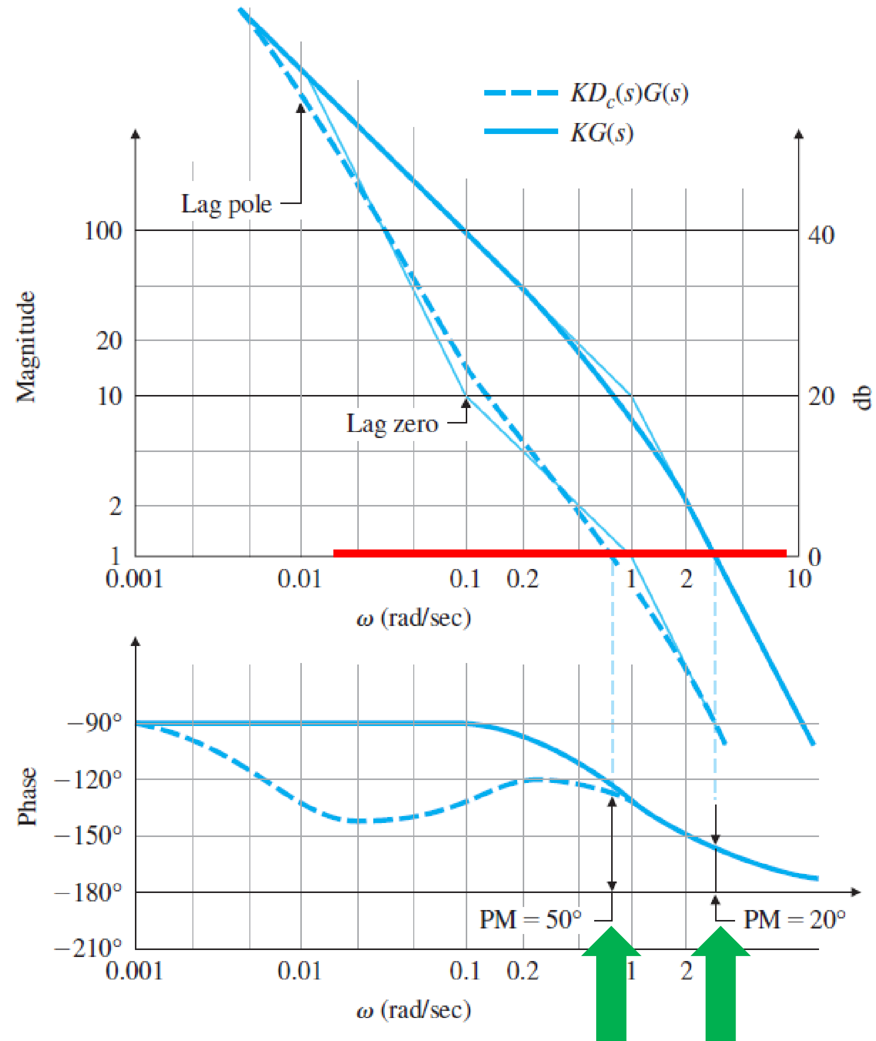


(b)

- Example 6.19: Lag Compensation for the DC Motor

$$G(s) = \frac{1}{s(s+1)}$$

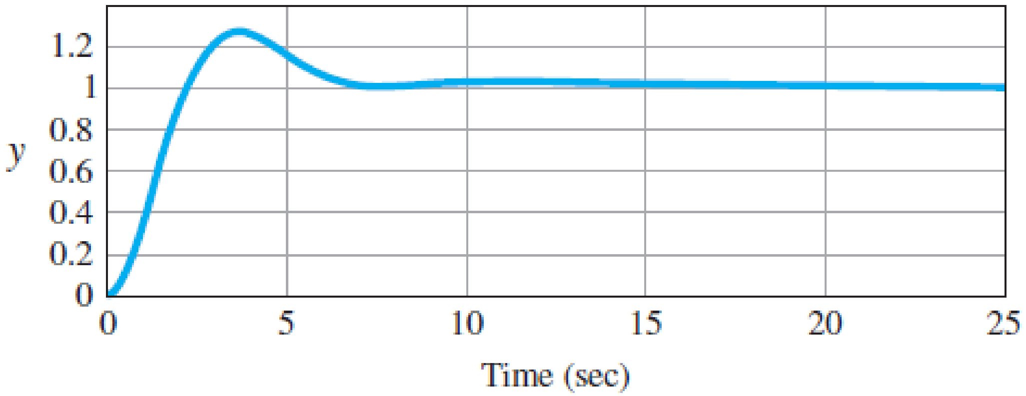
- Error constant: $K_v = 10$
- PM = 45^o
- K = 10
- PM = 20^o at ω_c ~= 3
- Select break points
  - ✓ ω_c is lowered
  - ✓ more favorable PM results
- ➢ Lag zero = 0.10
- ➢ Lag pole = 0.01
  - ✓ PM = 50^o



- Example 6.19: Lag Compensation for the DC Motor

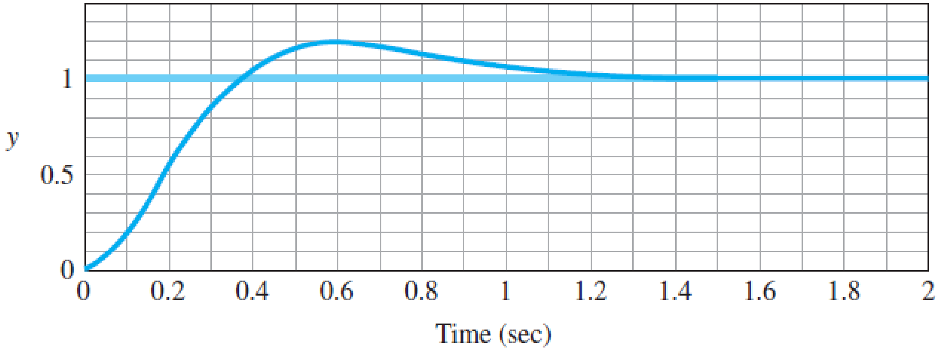
$$G(s) = \frac{1}{s(s+1)}$$

- Error constant: $K_v = 10$
- PM = 45^o



- No steady-state error
  - ✓ a Type 1 system
- Settling time ~= 25 sec
- Rise time ~= 2 sec

- Example 6.15: Lead Compensation



- Example 6.20: PID Compensation for Spacecraft Attitude Control

$$D_c(s) = \frac{K}{s}\left[(T_D s + 1)\left(s + \frac{1}{T_I}\right)\right]$$

$$G(s) = \frac{0.9}{s^2}$$

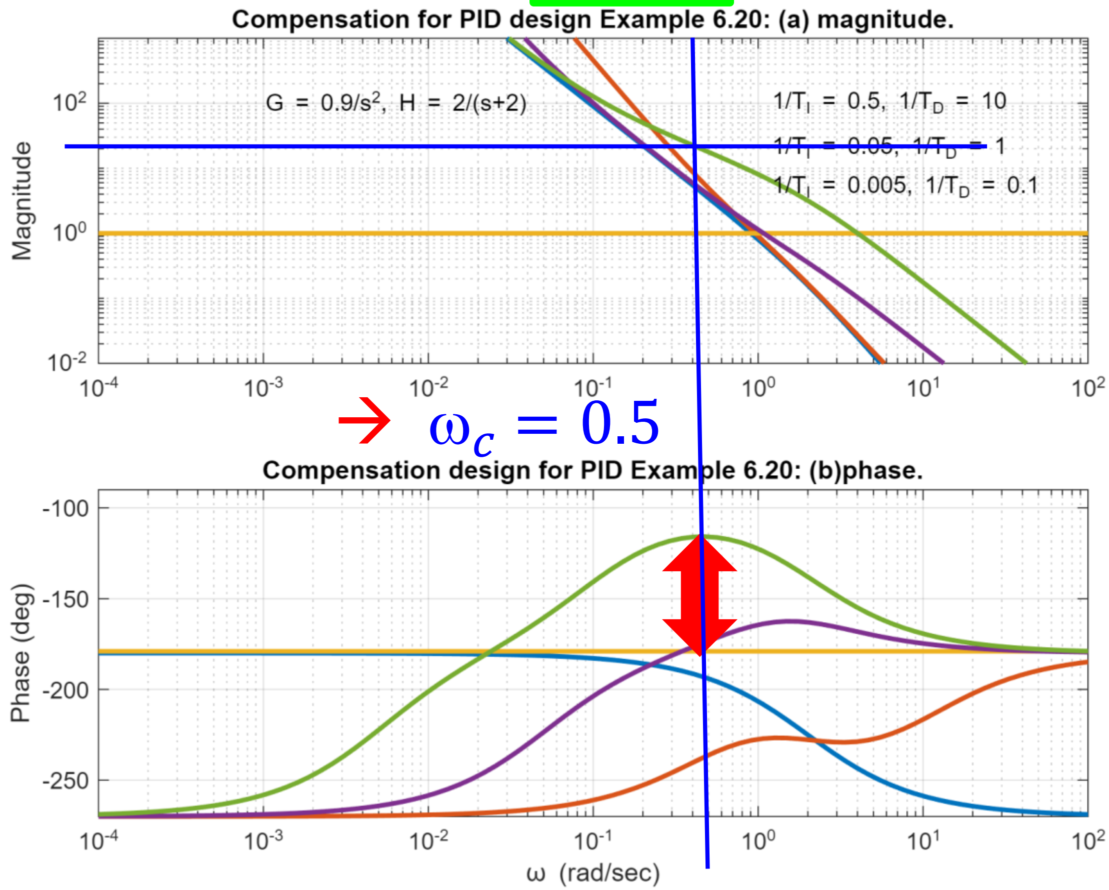
$$H(s) = \frac{2}{s+2}$$

$\frac{1}{T_I} = 0.5$ $\frac{1}{T_D} = 10$

$\frac{1}{T_I} = 0.05$ $\frac{1}{T_D} = 1$

$\frac{1}{T_I} = 0.005$ $\frac{1}{T_D} = 0.1$

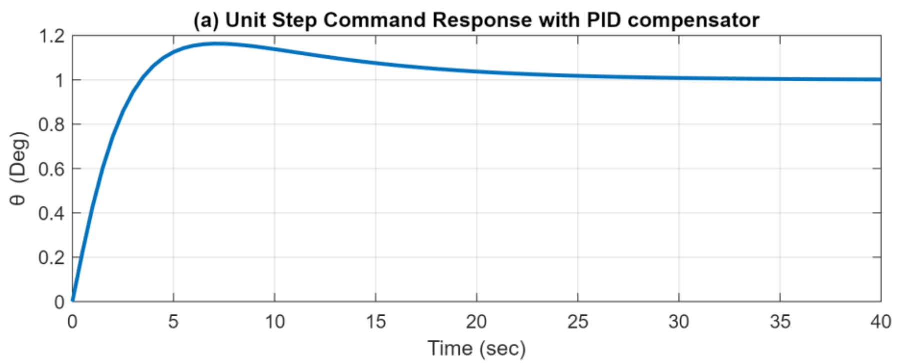
- PM = 65^o
- → Find K
- $|\ D_c(s)\, G(s)\ | = 20$
- 1/K = 20
- K = 0.05



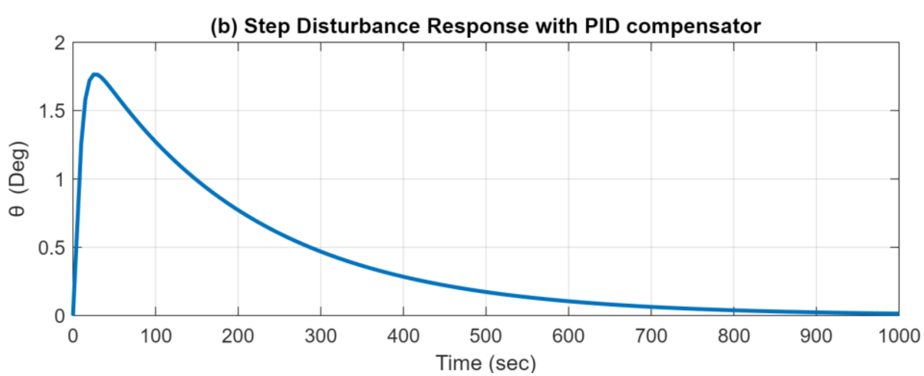
Example

- Example 6.20: PID Compensation for Spacecraft Attitude Control

$$\mathcal{T}(s) = \frac{\Theta}{\Theta_{com}} = \frac{D_c\, G}{1 + D_c\, G\, H}$$



$$\frac{\Theta}{T_d} = \frac{G}{1 + D_c\, G\, H}$$



- Frequency Response of $T(s)$ and $S(s)$ are shown: