Spring 2021

控制系統
Control Systems

Unit 4B
Control of Steady-State Error to Polynomial Inputs: System Type

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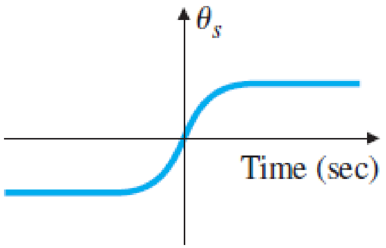
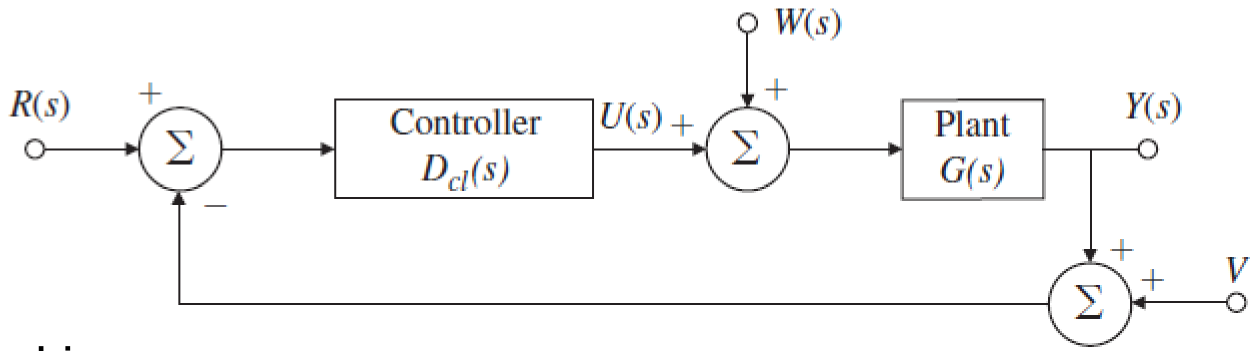
Control of Steady-State Error to Polynomial Inputs

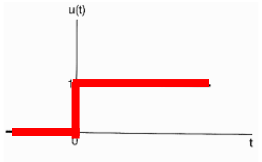
- Closed-loop system:



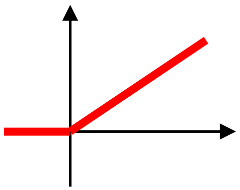


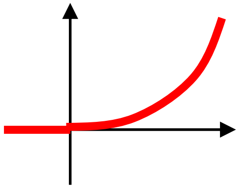
- Signal for satellite tracking

Regulation:



- Reference input is taken to be
  - A constant function,
  - Constant for a long periods of time,
  - A polynomial in time, of low degree.



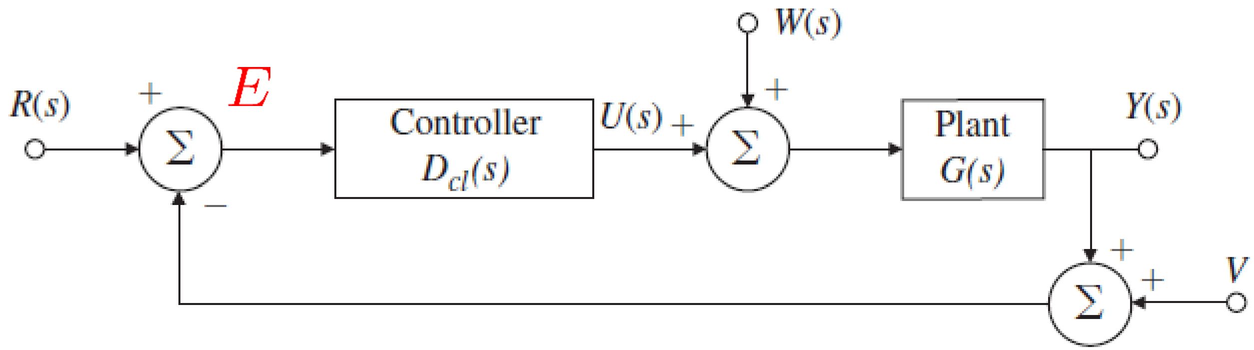


System Type:

- The degree of polynomial that they can reasonably track.

System Type for Tracking

- Closed-loop system:

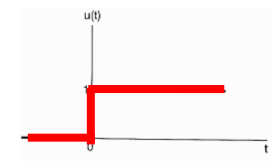


$$E = \frac{1}{1 + G\,D_{cl}}\,R \qquad \mathcal{S} = \frac{1}{1 + G\,D_{cl}}$$

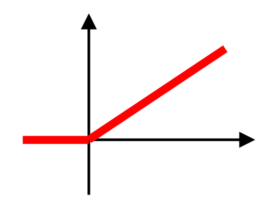
$$= \mathcal{S}\,R$$

$$r(t) = \frac{t^k}{k!}\,1(t)$$

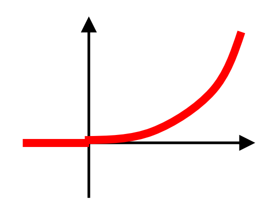
$$R(s) = \frac{1}{s^{k+1}}$$



- Consider polynomial inputs:



- k = 0: step input, position input
- k = 1: ramp input, velocity input
- k = 2: acceleration input



System Type for Tracking

- Steady-State Error by the Final-Value Theorem:

$$e_{ss} = \lim_{t\to\infty} e(t)$$

$$= \lim_{s\to 0} s\,E(s) \qquad = \lim_{s\to 0} s\,\frac{1}{1 + G\,D_{cl}}\,R$$

$$= \lim_{s\to 0} s\,\frac{1}{1 + G\,D_{cl}}\,\frac{1}{s^{k+1}}$$

- Type 0:
  - If *GD* has no pole at the origin (no integrator)
  - With a unit-step input:

$$r(t) = 1(t)$$

$$R(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s\to 0} s\,\frac{1}{1 + G\,D_{cl}}\,\frac{1}{s} = \frac{1}{1 + GD_{cl}(0)}$$

$$\frac{e_{ss}}{r_{ss}} = \frac{e_{ss}}{1} \qquad \Rightarrow e_{ss} = \frac{1}{1 + GD_{cl}(0)}$$

- Position Error Constant: $GD_{cl}(0) \triangleq K_p$

System Type for Tracking

- In general case:

$$GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n}$$

$GD_{cl}^0(s)$ has no pole at the origin

$$GD_{cl}^0(0) = K_n$$

- n = 0: GD_{cl} has no integrator
- n = 1: GD_{cl} has one integrator
- n = 2: GD_{cl} has two integrators

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1 + \frac{GD_{cl}^0(s)}{s^n}} \frac{1}{s^{k+1}}$$

$$= \lim_{s\to 0} s \frac{1}{\frac{s^n}{s^n} + \frac{GD_{cl}^0(s)}{s^n}} \frac{1}{s^{k+1}}$$

$$= \lim_{s\to 0} s \frac{s^n}{s^n + GD_{cl}^0(s)} \frac{1}{s^{k+1}} = \lim_{s\to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

System Type for Tracking

- In general case: $$e_{ss} = \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$
- If n > k, $$e_{ss} = 0$$
- If n < k, $$e_{ss} \to \infty$$
- If n = k = 0, $$e_{ss} = \frac{1}{1 + K_n}$$
- If n = k =/= 0, $$e_{ss} = \frac{1}{K_n}$$
- Type 0:
  - n = k = 0
  - a step (position) input
  - zero-degree polynomial
  - K_0: position constant, $K_0 = K_p$

System Type for Tracking

- In general case: $e_{ss} = \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

$$K_p = \lim_{s \to 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \to 0} s\, GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \to 0} s^2 GD_{cl}(s), \quad n = 2$$

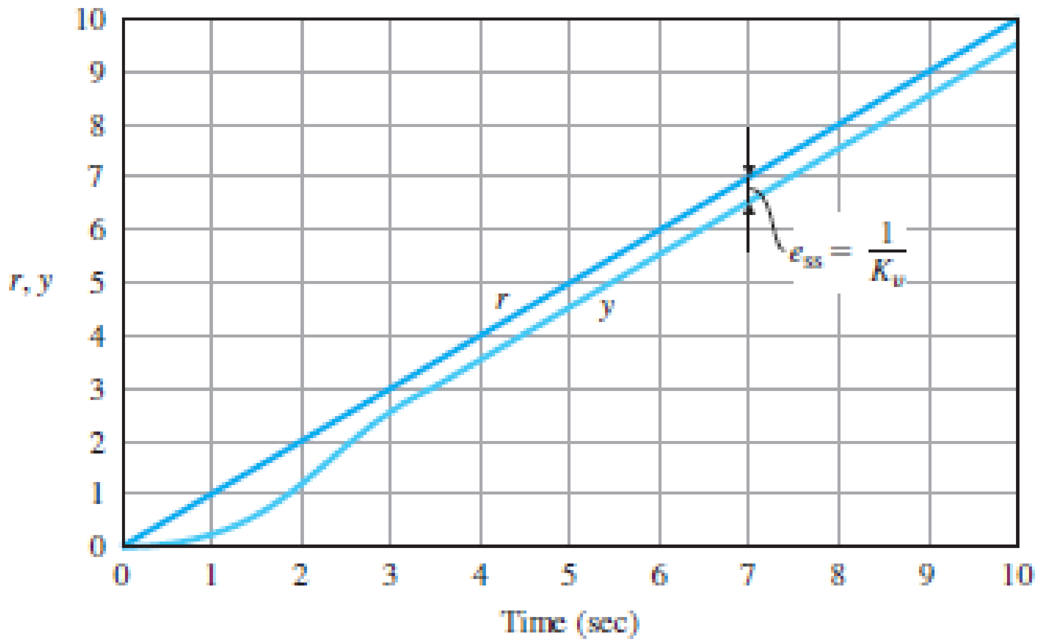
$$GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n}$$

- System Type: a robust property with respect to parameter changes in the unity feedback structure.

System Type for Tracking

- In general case: $$e_{ss} = \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

- Relationship between ramp response and K_v



Example 4.1: System Type for Speed Control

$$G(s) = \frac{A}{\tau s + 1}$$

$$D(s) = k_P$$

$$\Rightarrow GD_{cl}(s) = \frac{k_P A}{\tau s + 1}$$

$$\Rightarrow \text{no pole at } s = 0, \quad n = 0$$

$$\Rightarrow \text{Type 0}, \quad K_p = k_P A$$

$$K_p = \lim_{s \to 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \to 0} s\, GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \to 0} s^2\, GD_{cl}(s), \quad n = 2$$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1 + G\, D_{cl}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \to 0} s \frac{1}{1 + \frac{k_P A}{\tau s + 1}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \to 0} \frac{\tau s + 1}{\tau s + 1 + k_P A} \frac{1}{s^k}$$

$$k = 0, \quad e_{ss} = \frac{1}{1 + k_P A}$$

$$= \frac{1}{1 + K_p}$$

$$k = 1, \quad e_{ss} = \infty$$

▪ Example 4.2: System Type using Integral Control

$$G(s) = \frac{A}{\tau s + 1}$$

$$D(s) = k_P + \boxed{\frac{k_I}{s}}$$

$$\Rightarrow GD_{cl}(s) = \frac{A(k_P s + k_I)}{s(\tau s + 1)}$$

$$K_p = \lim_{s\to 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s\to 0} s\, GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s\to 0} s^2\, GD_{cl}(s), \quad n = 2$$

$\Rightarrow$ single pole at $s = 0$, $n = 1$

$\Rightarrow$ Type 1, $K_v = k_I A$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1 + G D_{cl}} \frac{1}{s^{k+1}}$$

$$= \lim_{s\to 0} s \frac{1}{1 + \frac{A(k_P s + k_I)}{s(\tau s + 1)}} \frac{1}{s^{k+1}}$$

$$= \lim_{s\to 0} \frac{s(\tau s + 1)}{s(\tau s + 1) + A(k_P s + k_I)} \frac{1}{s^k}$$

$k = 0$, $e_{ss} = 0$

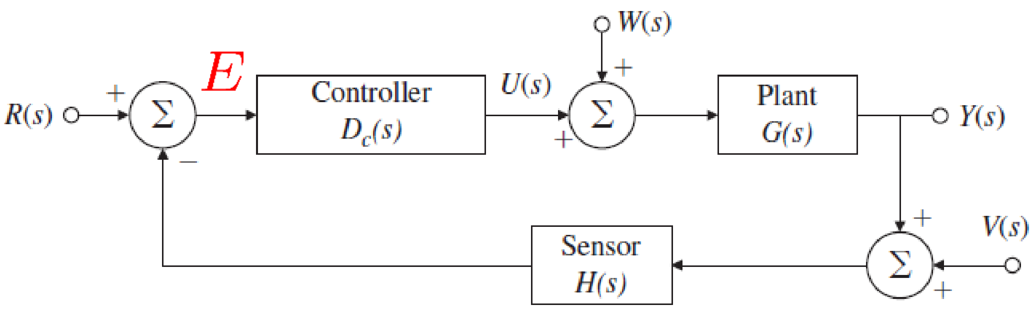
$k = 1$, $e_{ss} = \frac{1}{k_I A} = \frac{1}{K_v}$

$k = 2$, $e_{ss} = \infty$

From Closed-Loop Transfer Function

- Closed-loop system with sensor dynamics.

  R = reference, U = control, Y = output, V = sensor noise



$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{G\,D_c}{1 + G\,D_c\,H}$$

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s)\,R(s) = R(s)\,[\,1 - \mathcal{T}(s)\,]$$

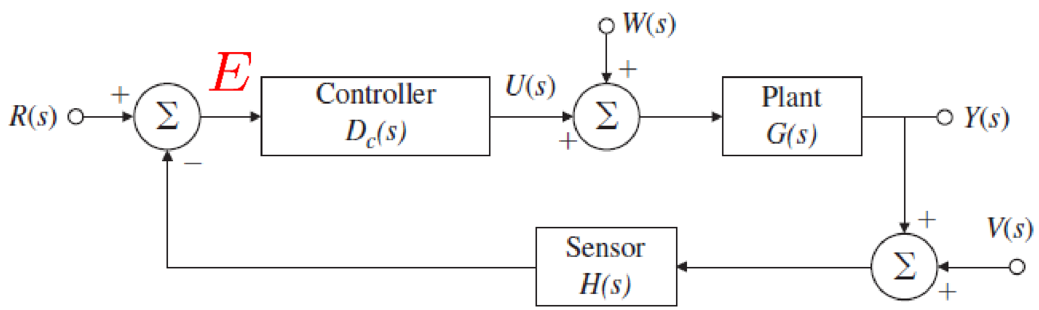
$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\,[\,1 - \mathcal{T}(s)\,]\,R(s)$$

- If the reference input a polynomial of degree k:

$$E(s) = \frac{1}{s^{k+1}}\,[\,1 - \mathcal{T}(s)\,]$$

$$e_{ss} = \lim_{s\to 0} s\,\frac{1 - \mathcal{T}(s)}{s^{k+1}} = \lim_{s\to 0} \frac{1 - \mathcal{T}(s)}{s^{k}}$$

- Example 4.3: System Type for Servo w/ Tachometer Feedback



$$G(s) = \frac{1}{s(\tau s + 1)}$$

$$D_c(s) = k_P$$

$$H(s) = 1 + k_t s$$

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s) R(s)$$

$$= R(s) - \frac{G D_c}{1 + H G D_c} R(s) = \frac{1 + (H-1) G D_c}{1 + H G D_c} R(s)$$

$$e_{ss} = \lim_{s \to 0} s R(s) [1 - \mathcal{T}(s)] \qquad R(s) = \frac{1}{s^{k+1}}$$

$$= \lim_{s \to 0} \frac{1 - \mathcal{T}(s)}{s^k} = \lim_{s \to 0} \frac{1}{s^k} \frac{s(\tau s + 1) + (1 + k_t s - 1) k_P}{s(\tau s + 1) + (1 + k_t s) k_P}$$

$$= 0 \qquad k = 0$$

$$= \frac{1 + k_t k_P}{k_P}, \qquad k = 1$$

System Type for Regulation and Disturbance Rejection

- Transfer function from disturbance input W(s) to error E(s)

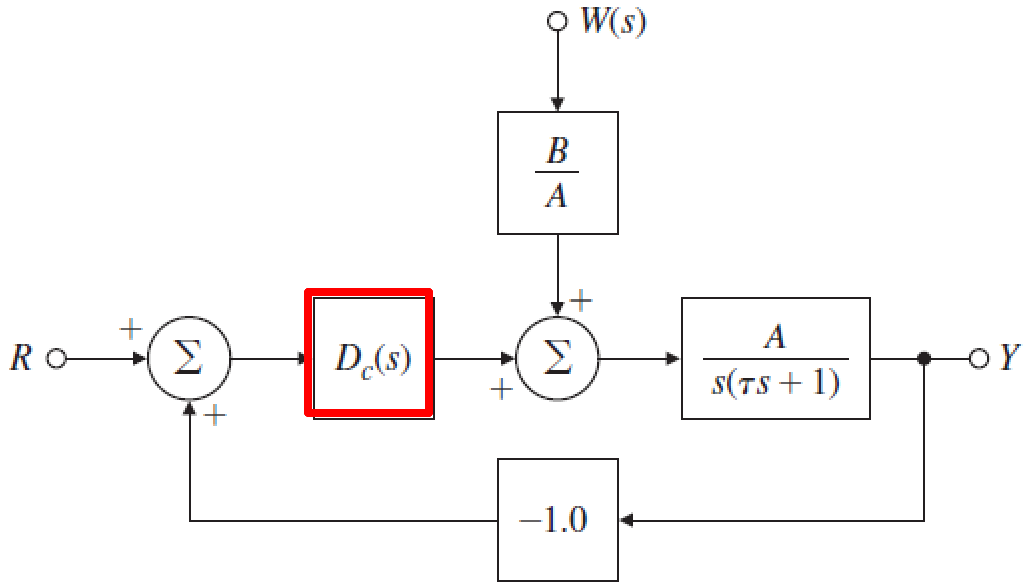
$$\frac{E(s)}{W(s)} = -\frac{Y(s)}{W(s)} = \mathcal{T}_w(s) \quad \text{with } R(s) = 0$$

$$\mathcal{T}_w(s) = s^n \, \mathcal{T}_{w,0}(s) \quad \text{with } \mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$$

$$y_{ss} = \lim_{s \to 0} s \, \mathcal{T}_w(s) \frac{1}{s^{k+1}} = \lim_{s \to 0} \mathcal{T}_{w,0}(s) \frac{s^n}{s^k}$$

- If n > k, $y_{ss} = 0$
- If n < k, $y_{ss} \to \infty$
- If n = k, $y_{ss} = \frac{1}{K_{w,n}}$

- Example 4.4: System Type for DC Motor Position Control

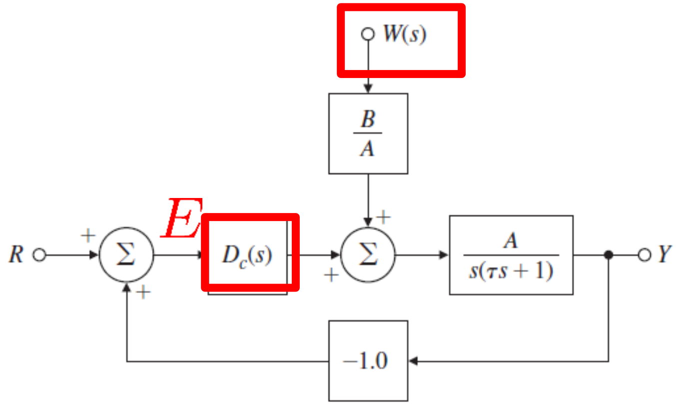


(a) $D_c(s) = k_P$

(b) $D_c(s) = k_P + \dfrac{k_I}{s}$

- Example 4.4: System Type for DC Motor Position Control

(a) $D_c(s) = k_P$



- Transfer function from W to E (R=0)

$$\mathcal{T}_w(s) = -\frac{B}{s(\tau s + 1) + A\,k_P}$$

$$= s^0\,\mathcal{T}_{w,0}(s)$$

$n = 0$ $\quad \mathcal{T}_w(s) = s^n\,\mathcal{T}_{w,0}(s)$

$$K_{w,0} = -\frac{A\,k_P}{B}$$

with $\mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$

- Type 0: Steady-state error to a unit-step torque input is:

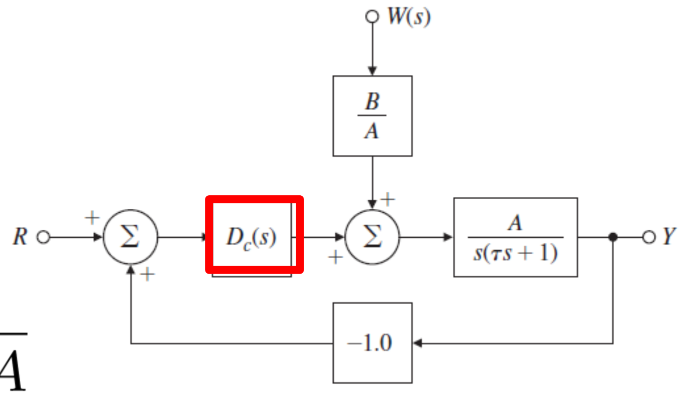
$$e_{ss} = -\frac{B}{A\,k_P}$$

$$y_{ss} = \frac{1}{K_{w,n}}$$

- Steady-state error to a unit-ramp torque input is: $e_{ss} = -\infty$

- Example 4.4: System Type for DC Motor Position Control

(b) $D_c(s) = k_P + \frac{k_I}{s}$



- Transfer function from W to E (R=0)

$$\mathcal{T}_w(s) = -\frac{Bs}{s^2(\tau s + 1) + (k_P\, s + k_I\,)A}$$

$$n = 1 \qquad \mathcal{T}_w(s) = s^n\, \mathcal{T}_{w,0}(s)$$

$$K_{w,n} = -\frac{A\,k_I}{B} \qquad \text{with} \quad \mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$$

- Type 1: Steady-state error to a unit-ramp disturbance input is:

$$e_{ss} = -\frac{B}{A\,k_I} \qquad y_{ss} = \frac{1}{K_{w,n}}$$

- Steady-state error to a unit t^2 disturbance input is: $e_{ss} = -\infty$
- Steady-state error to a unit-step disturbance input is: $e_{ss} = 0$

Summary: System Type

- In general case: $e_{ss} = \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

$$K_p = \lim_{s \to 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \to 0} s\, GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \to 0} s^2\, GD_{cl}(s), \quad n = 2$$

$$GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n}$$

- System Type: a robust property with respect to parameter changes in the unity feedback structure.