

Spring 2021

控制系統
Control Systems

Unit 3C
Effect of Pole Locations

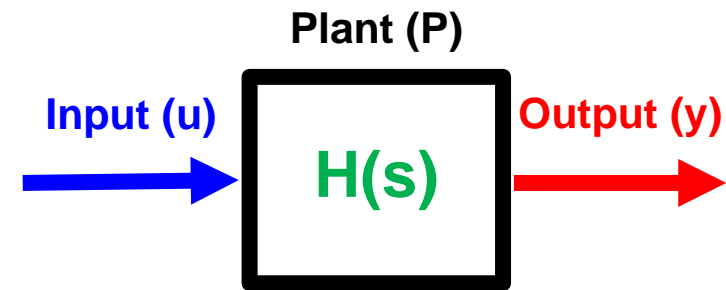
Feng-Li Lian

NTU-EE

Feb – Jun, 2021

- **Transfer Function** = a ration of polynomials:

$$H(s) = \frac{b(s)}{a(s)}$$

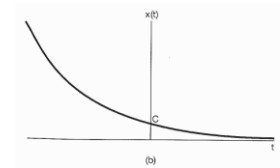


- $a(s)$ and $b(s)$ have **NO** common factors
- **Poles**: roots of $a(s) = 0$
- **Zores**: roots of $b(s) = 0$

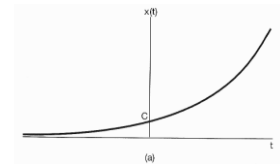
- Transfer Function and Impulse Response:

$$H(s) = \frac{1}{s + a} \Rightarrow h(t) = e^{-at} \mathbf{1}(t)$$

- $a > 0$: pole: $s = -a$, at **negative** s ,
the exponential expression **decays**
the impulse response is **stable**
- $a < 0$: pole: $s = -a$, at **positive** s ,
the exponential expression **grows**
the impulse response is **unstable**



$$\mathbf{1}(t), t \geq 0$$



$$\mathbf{1}(t), t \geq 0$$

- The **impulse response** is the **natural response** of the system

- Transfer Function, Impulse Response, Time Constant

$$H(s) = \frac{1}{s + a} \quad \Rightarrow \quad \tau \triangleq \frac{1}{a}$$

$$\Rightarrow h(t) = e^{-at} \mathbf{1}(t) \quad \text{at } t = \tau \quad \Rightarrow \quad h(\tau) = \frac{1}{e}$$

- Impulse Response:

$$u(t) = \delta(t) \quad \Rightarrow \quad U(s) = 1$$

$$Y(s) = H(s) U(s) = \frac{1}{s + a} \cdot 1 = \frac{1}{s + a}$$

$$\Rightarrow y(t) = e^{-at} \mathbf{1}(t)$$

- Step Response:

$$u(t) = \mathbf{1}(t) \quad \Rightarrow \quad U(s) = \frac{1}{s}$$

$$Y(s) = H(s) U(s) = \frac{1}{s + a} \cdot \frac{1}{s} = \frac{1}{a} \left(\frac{1}{s} - \frac{1}{s + a} \right)$$

$$\Rightarrow y(t) = \frac{1}{a} (1 - e^{-at}) \mathbf{1}(t)$$

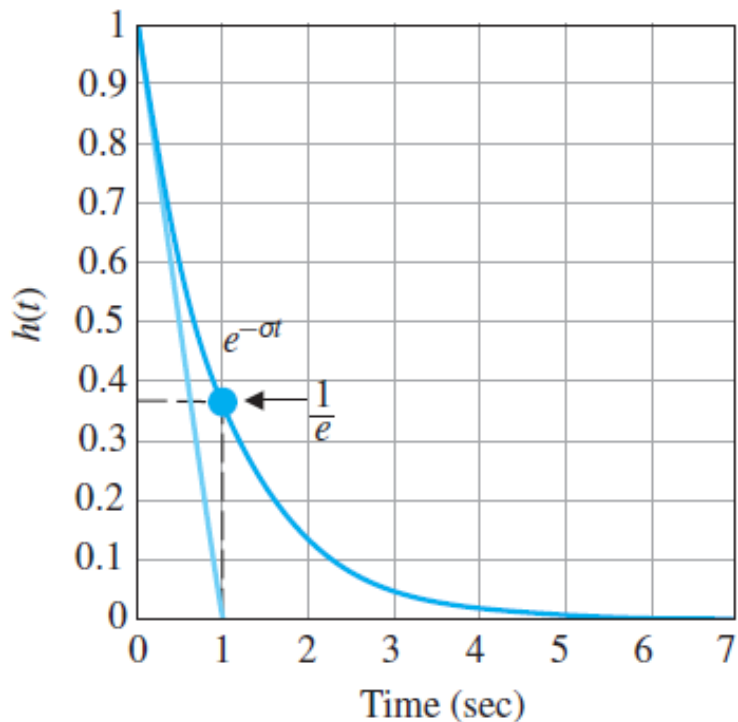
Effect of Pole Locations

- Transfer Function, Impulse Response, Time Constant

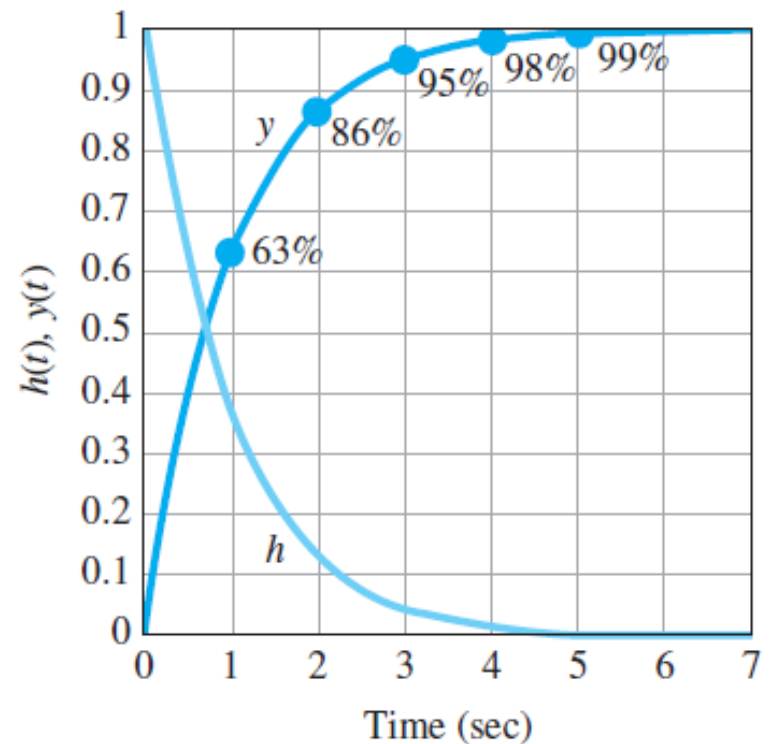
$$H(s) = \frac{1}{s + a} \quad \Rightarrow \quad h(t) = e^{-at} \mathbf{1}(t)$$

- Impulse Response: $\Rightarrow y(t) = e^{-at} \mathbf{1}(t)$

- Step Response: $\Rightarrow y(t) = \frac{1}{a}(1 - e^{-at}) \mathbf{1}(t)$



(a)



(b)

Example 3.25: Response vs Pole Locations, Real Roots

$$H(s) = \frac{2s + 1}{s^2 + 3s + 2}$$

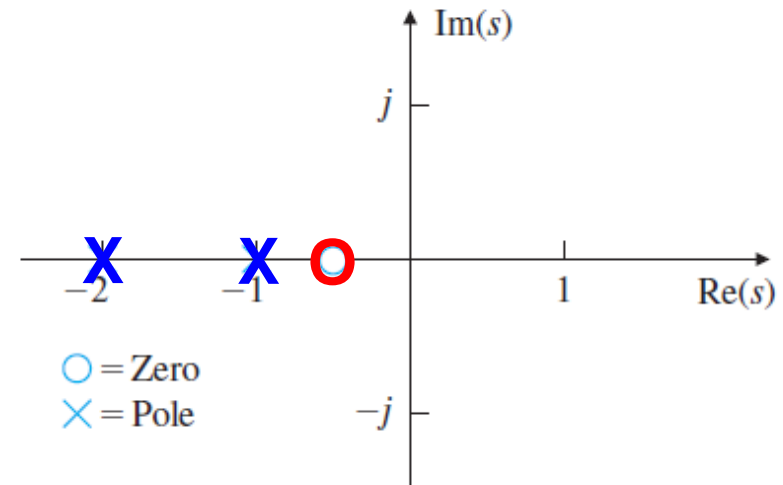
- poles as crosses (x) and zeros as circles (o)

$$a(s) = s^2 + 3s + 2 = 0$$

$$\text{Poles: } s = -1, -2$$

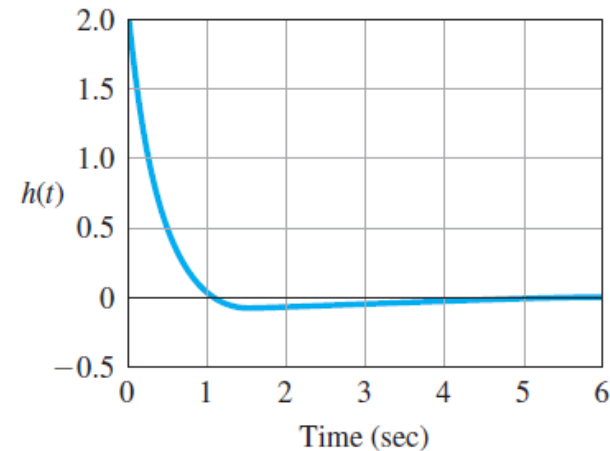
$$b(s) = 2s + 1 = 0$$

$$\text{Zeros: } s = -1/2$$

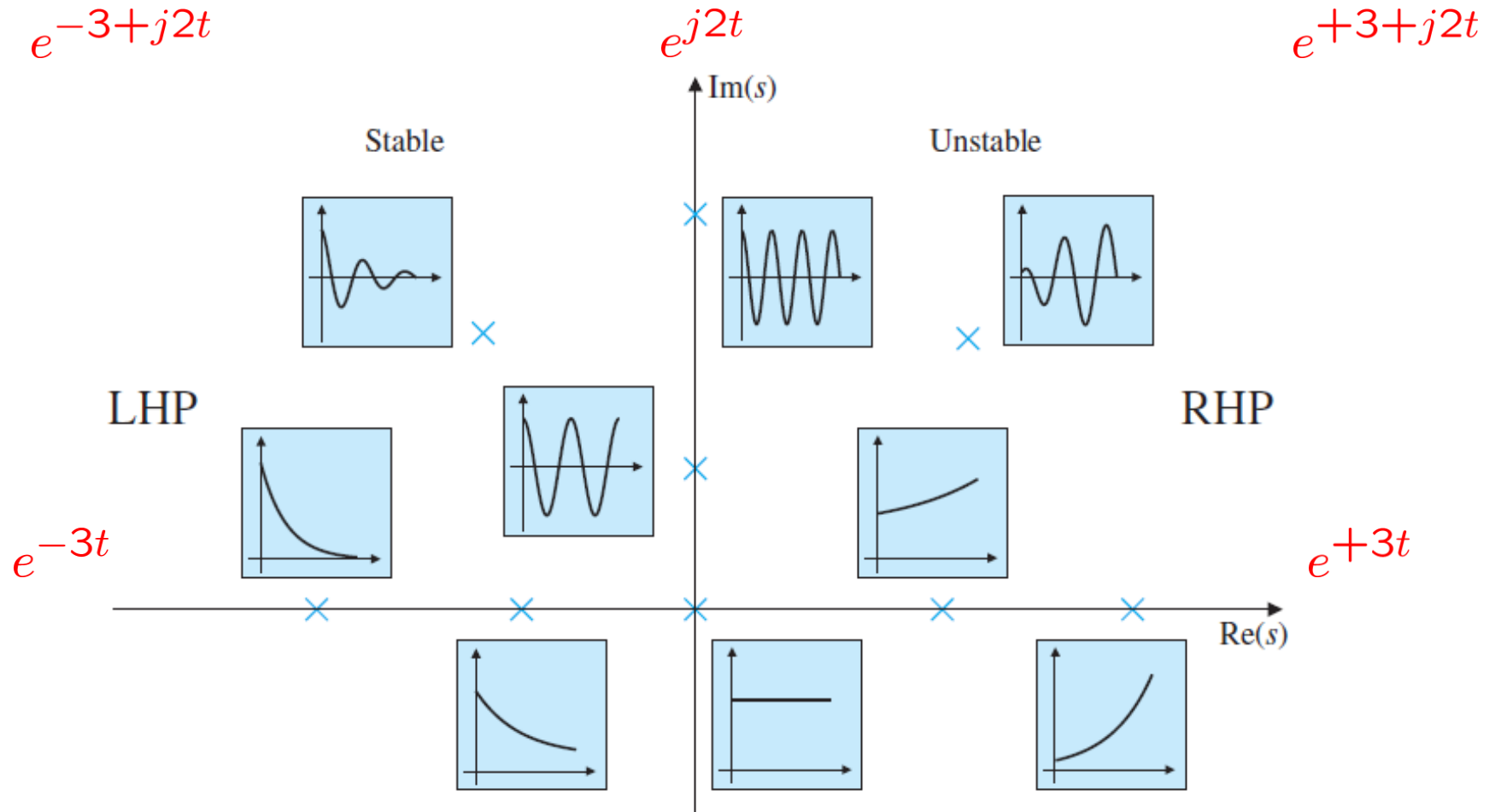


$$H(s) = -\frac{1}{s + 1} + \frac{3}{s + 2}$$

$$\Rightarrow h(t) = (-e^{-t} + 3e^{-2t}) 1(t)$$



- Time functions associated with points in the s-plane
(LHP, left half-plane; RHP, right half-plane)



- Complex Poles $s = -\sigma \pm j\omega_d$

$$a(s) = (s + \sigma - j\omega_d)(s + \sigma + j\omega_d) = (s + \sigma)^2 + \omega_d^2$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

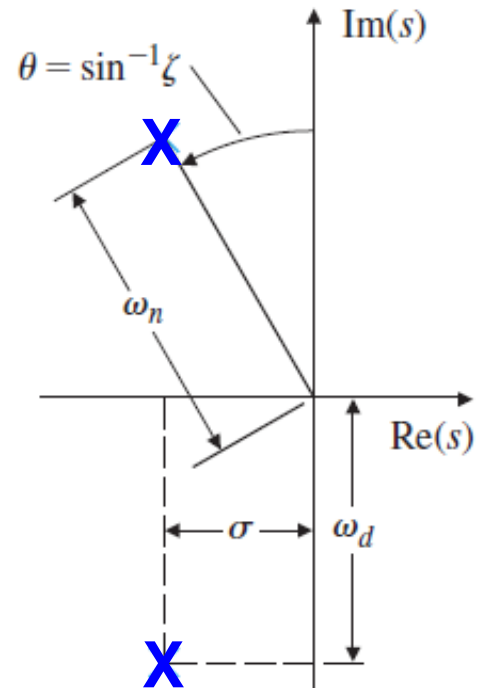
$$\sigma = \omega_n \zeta$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

- Damping Ratio: ζ
- Undamped Natural Frequency: ω_n

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} (\sin \omega_d t) 1(t)$$



- s-plane plot for a pair of complex poles

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

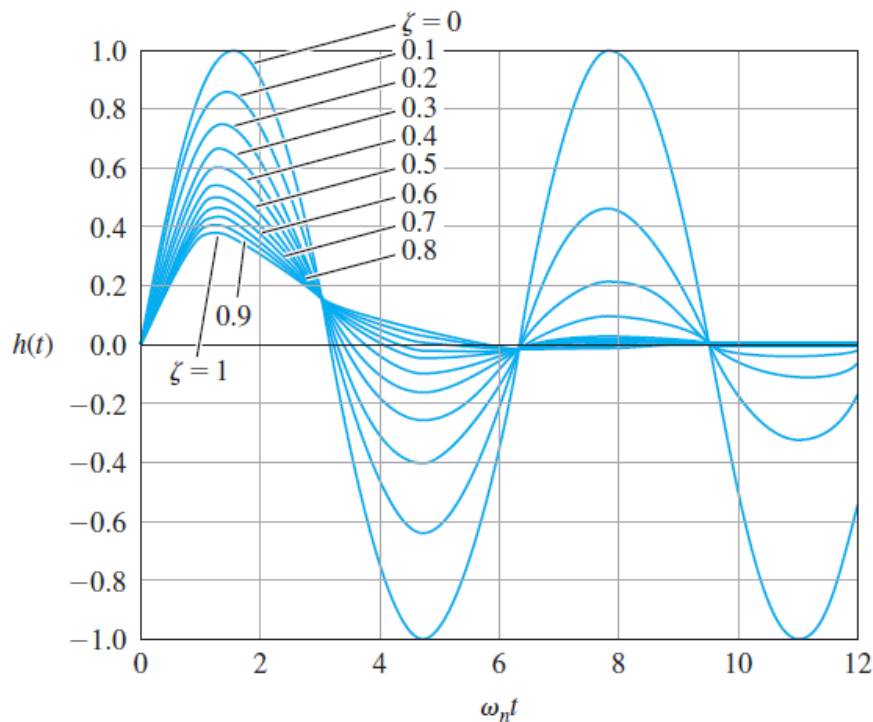
$$\sigma = \omega_n \zeta$$

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} (\sin \omega_d t) 1(t)$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

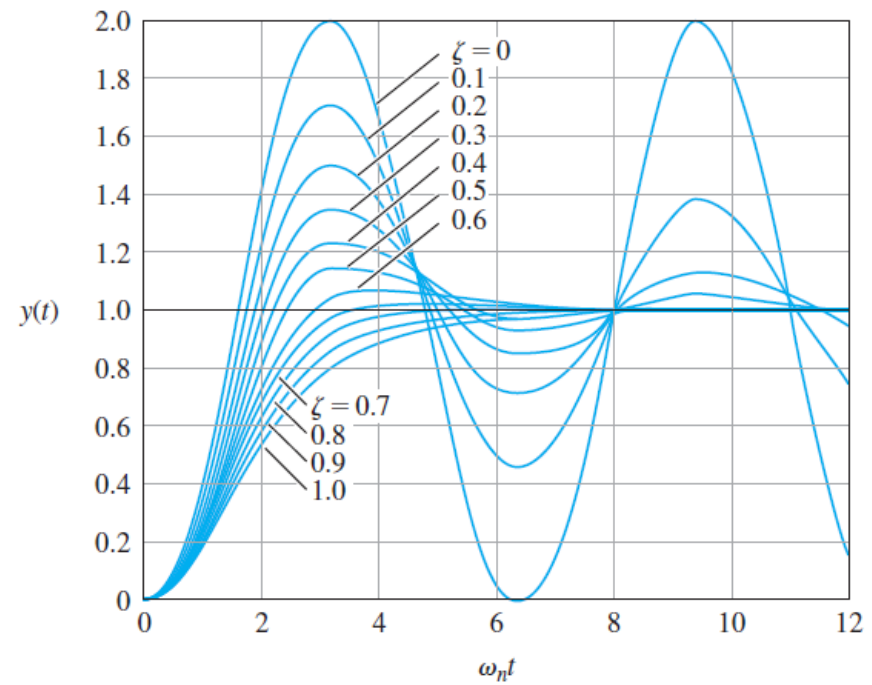
- Responses of second-order systems versus ζ :

(a) Impulse Responses



(a)

(b) Step Responses



(b)

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$\sigma = \omega_n \zeta$$

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} (\sin \omega_d t) 1(t)$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

```
s = tf( 's' )
```

```
wn = 1;
```

```
zeta = 0.3;
```

```
sysH = wn^2/( s^2 + 2*zeta*wn*s + wn^2 );
```

```
t = 0:0.01:10;
```

```
y1 = step( sysH, t );
```

```
y2 = impulse( sysH, t );
```

```
figure(1)
```

```
subplot(2,1,1)
```

```
plot( t, y1 )
```

```
grid
```

```
axis( [ 0, 10, -0.1, 2] )
```

```
title('step');
```

```
hold on;
```

```
subplot(2,1,2)
```

```
plot( t, y2 )
```

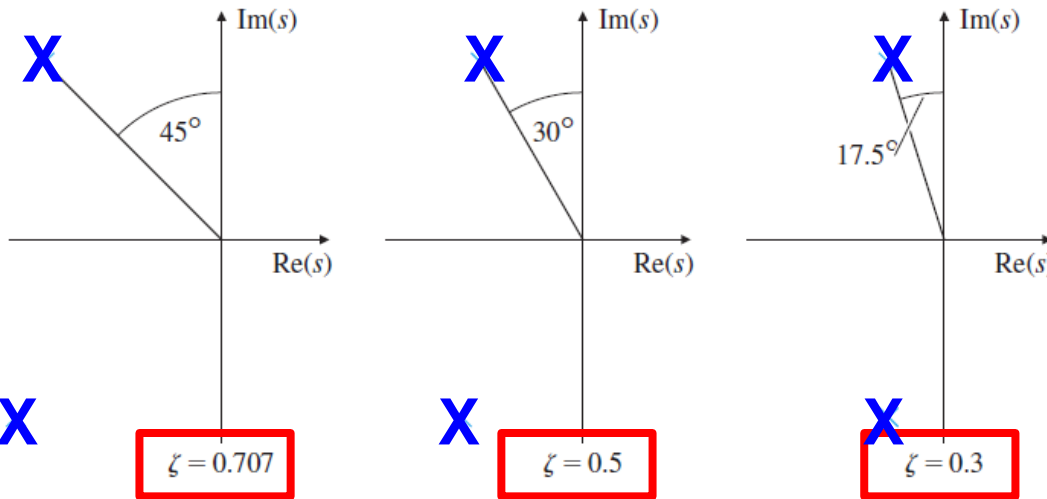
```
grid
```

```
axis( [ 0, 10, -2, 2] )
```

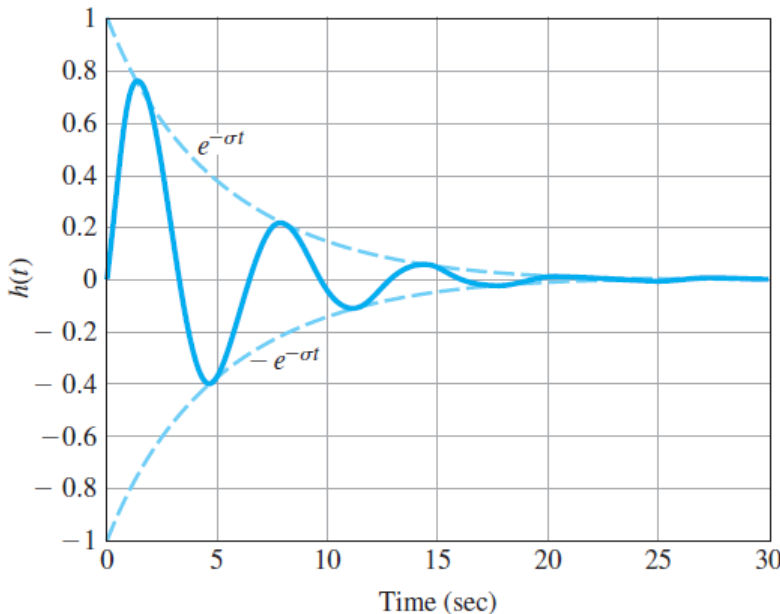
```
title('impulse');
```

```
hold on;
```

- Stability depends on whether natural response grows or decays



- Pole locations corresponding to three values of ζ



$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} (\sin \omega_d t) 1(t)$$

- Second-order system response with an exponential envelope bound

■ Example 3.26: Oscillatory Time Response

$$H(s) = \frac{2s + 1}{s^2 + 2s + 5} = \frac{2s + 1}{(s + 1)^2 + 2^2}$$

$$\omega_n^2 = 5 \quad \omega_n = \sqrt{5} = 2.24 \text{ rad/sec}$$

$$2 \zeta \omega_n = 2 \quad \zeta = \frac{1}{\sqrt{5}} = 0.447$$

$$H(s) = 2 \frac{s + 1}{(s + 1)^2 + 2^2} - \frac{1}{2} \frac{2}{(s + 1)^2 + 2^2}$$

$$\Rightarrow h(t) = \left(2 e^{-t} \cos 2t - \frac{1}{2} e^{-t} \sin 2t; \right) 1(t)$$

Example 3.26: Oscillatory Time Response

```
s = tf( 's' )
```

```
sysH = ( 2*s + 1 ) / ( s^2 + 2*s + 5 );
```

```
t = 0:0.1:6;
```

```
y = impulse( sysH, t );
```

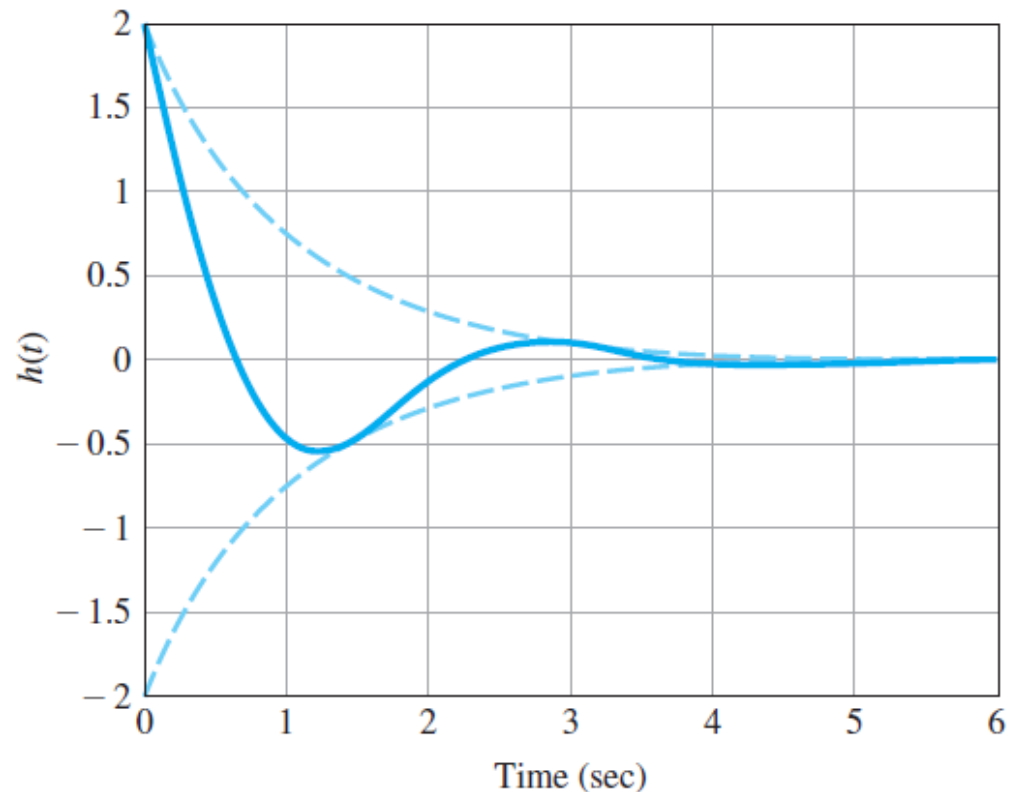
```
plot( t, y )
```

```
grid
```

```
hold on;
```

```
plot( t, 2*exp(-t), ':' )
```

```
plot( t, -2*exp(-t), ':' )
```



● system response