

Spring 2020

控制系統
Control Systems

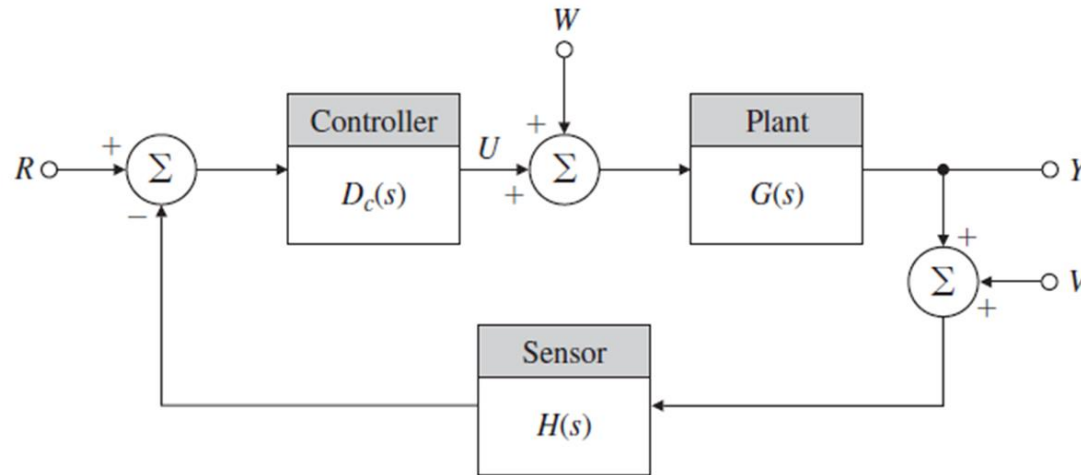
Unit 51
Root Locus of a Basic Feedback System

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- Basic Feedback System:



- Closed-Loop Transfer Function:

$$\mathcal{T}(s) = \frac{Y(s)}{R(s)} = \frac{D_c(s) G(s)}{1 + D_c(s) G(s) H(s)}$$

- The Characteristic Equation:

$$1 + D_c(s) G(s) H(s) = 0 \quad \Rightarrow \quad 1 + K \frac{b(s)}{a(s)} = 0$$

- The roots are the poles of the transfer function $\Rightarrow a(s) + K b(s) = 0$

Example 0

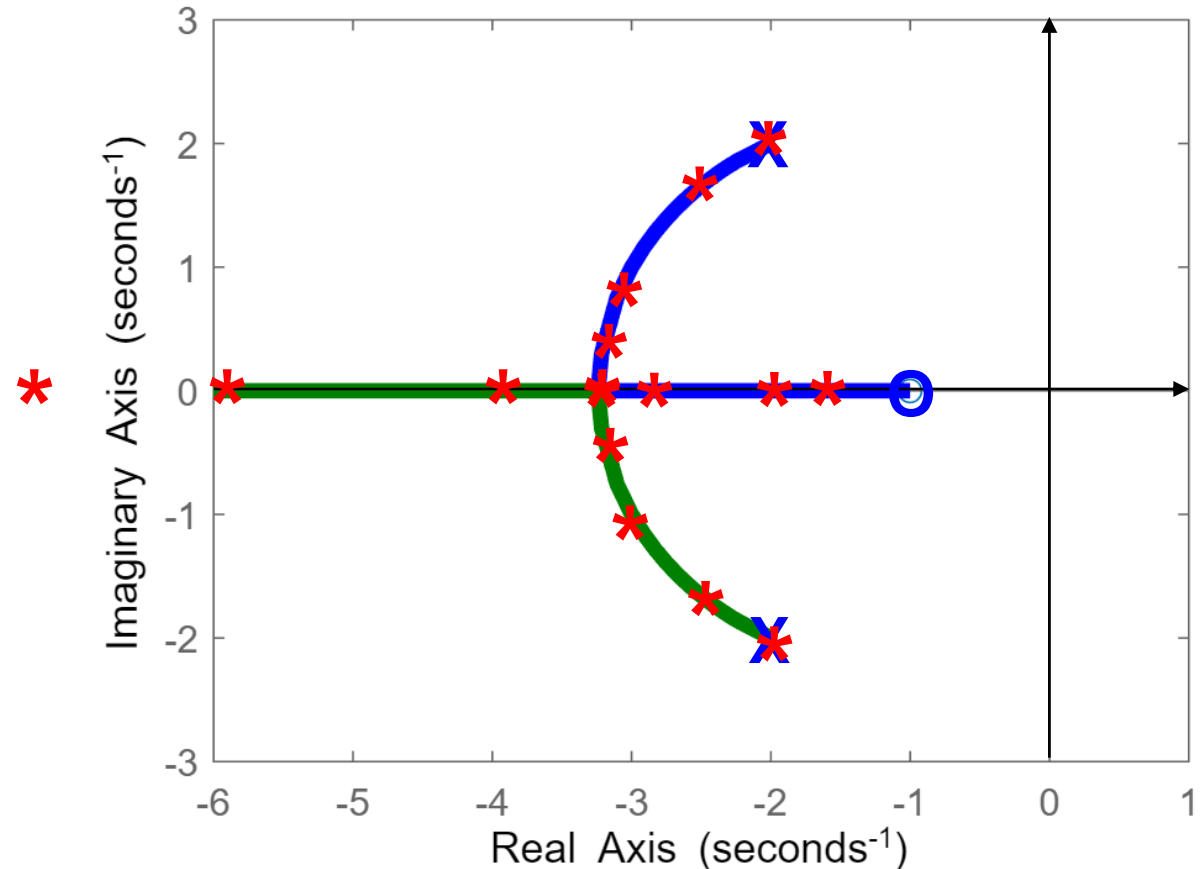
$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0 \quad \Rightarrow a(s) + K b(s) = 0$$

$$\Rightarrow 1 + K \frac{s + 1}{s^2 + 4s + 8} = 0 \quad \Rightarrow s^2 + 4s + 8 + K(s + 1) = 0$$

$$\Rightarrow s^2 + (4 + K)s + (8 + K) = 0$$

K	roots
0	-2 ± j(2)
1	-2.5 ± j(1.66)
2	-3 ± j(1)
2.4	-3.2 ± j(0.4)
2.47	-3.24 ± j(0.3)
2.6	-3.84 ± j(0.2)
3	-4.62 ± j(0.1)
4	-6 ± j(0)
5	-7.19 ± j(0)

Root Locus vs. K



- Root Locus:

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0 \quad \Rightarrow a(s) + K b(s) = 0$$

- K : From 0 to infinity

- The plot of all possible roots

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

$$= (s - p_1) (s - p_2) \dots (s - p_{n-1}) (s - p_n) = \prod_{i=1}^n (s - p_i)$$

$$b(s) = s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_{m-1} s + b_m$$

$$= (s - z_1) (s - z_2) \dots (s - z_{m-1}) (s - z_m) = \prod_{i=1}^m (s - z_i)$$

- For simplicity, assume that $n \geq m$,

- That is, it is a rational transfer function

$$\Rightarrow a(s) + K b(s) = (s - r_1) (s - r_2) \dots (s - r_{n-1}) (s - r_n)$$

- The **Root-Locus Form or Evan Form** of a Characteristic Equation:

$$1 + D_c(s) G(s) H(s) = 0$$

$$\Rightarrow 1 + K L(s) = 0$$

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0$$

$$\Rightarrow a(s) + K b(s) = 0$$

$$\Rightarrow L(s) = -\frac{1}{K}$$

- The **Root-Locus Method**:

a method for inferring **dynamic properties** of the closed-loop systems
as **K** changes

Examples

Example 5.1: Root Locus of a Motor Position Control

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{Y(s)}{U(s)} = G(s) = \frac{A}{s(s+c)}$$

$$D_c(s) = H(s) = 1, \quad c = 1$$

$$\Rightarrow L(s) = \frac{b(s)}{a(s)} = \frac{1}{s(s+1)} \Rightarrow \begin{cases} m = 0 & z_i = \{ \text{empty} \} \\ n = 2 & p_i = \{0, -1\} \end{cases}$$

$$\Rightarrow K = A$$

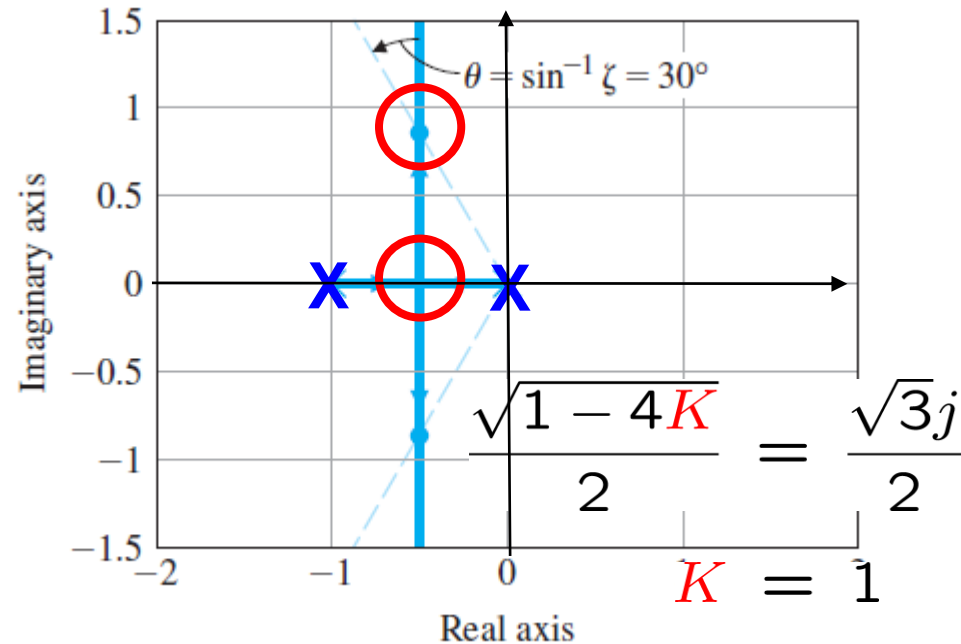
$$\Rightarrow a(s) + K b(s) = s^2 + s + K = 0$$

$$\Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{1 - 4K}}{2}$$

$$\Rightarrow 1 - 4K = 0$$

$$\Rightarrow r_{1,2} = -\frac{1}{2}, -\frac{1}{2}$$

$$\Rightarrow K = \frac{1}{4} \quad \blacksquare \text{ Breakaway point}$$



Examples

Example 5.2: Root Locus w.r.t. a Plant Open-Loop Pole

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{Y(s)}{U(s)} = G(s) = \frac{1}{s(s+c)}$$

$$1 + G(s) = 1 + \frac{1}{s(s+c)} \Rightarrow s^2 + cs + 1 = 0$$

$$\Rightarrow L(s) = \frac{b(s)}{a(s)} = \frac{s}{s^2 + 1} \Rightarrow \begin{cases} m = 1 & z_i = \{0\} \\ n = 2 & p_i = \{+j, -j\} \end{cases}$$

$$\Rightarrow K = c$$

$$\Rightarrow 1 + c \frac{s}{s^2 + 1} = 0$$

$$\Rightarrow r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4}}{2}$$

$$\Rightarrow c = 2 \quad \blacksquare \text{ Break-in point}$$

$$\Rightarrow r_{1,2} = -1, -1$$

