

Spring 2020

控制系統  
Control Systems

Unit 42

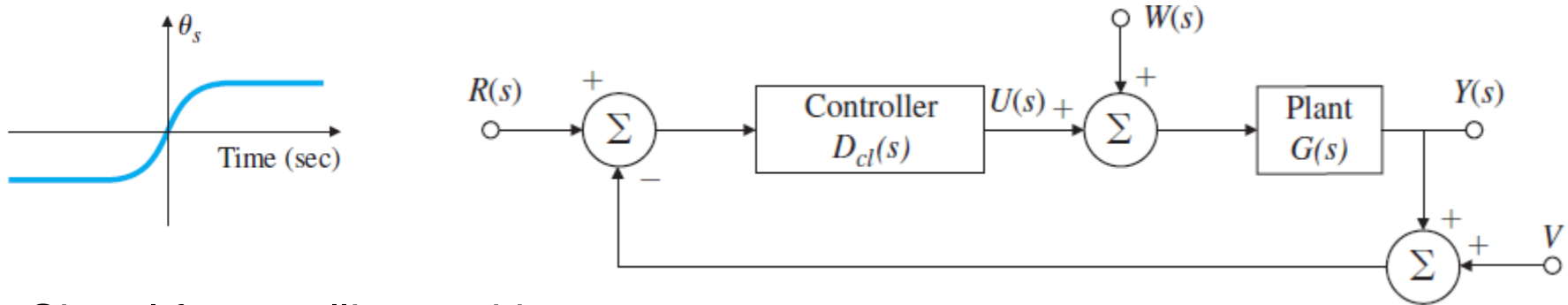
Control of Steady-State Error to Polynomial Inputs:  
System Type

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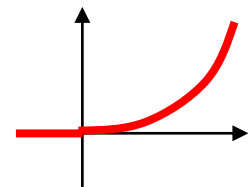
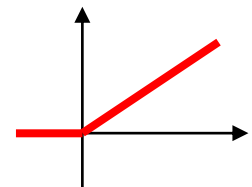
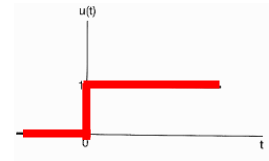
- Closed-loop system:



- Signal for satellite tracking

- Regulation:

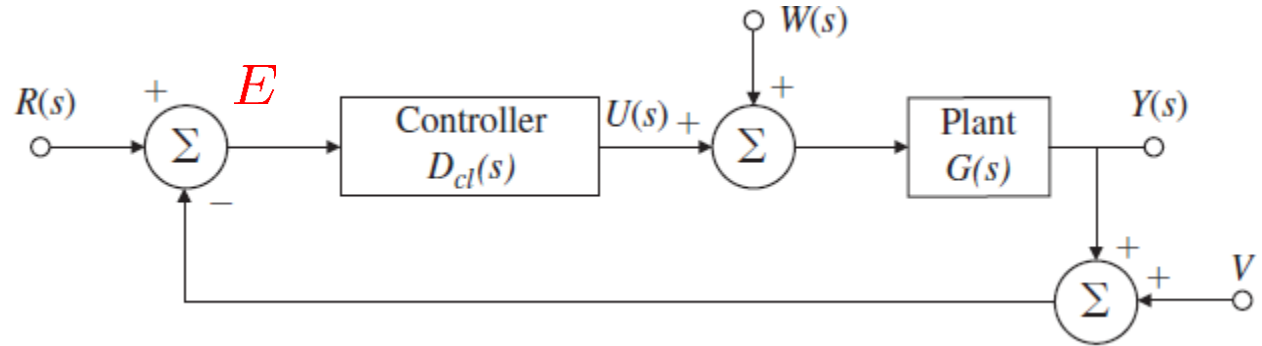
- Reference input is taken to be
  - A constant function,
  - Constant for a long periods of time,
  - A polynomial in time, of low degree.



- System Type:

- The degree of polynomial that they can reasonably track.

● Closed-loop system:



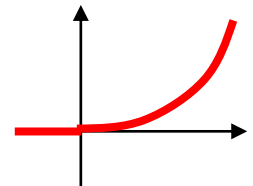
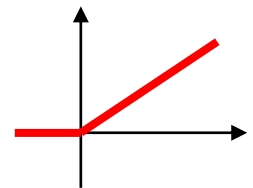
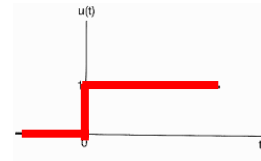
$$E = \frac{1}{1 + G D_{cl}} R$$

$$S = \frac{1}{1 + G D_{cl}}$$

$$= S R$$

$$r(t) = \frac{t^k}{k!} 1(t)$$

$$R(s) = \frac{1}{s^{k+1}}$$



■ Consider polynomial inputs:

■  $k = 0$ : step input, position input

■  $k = 1$ : ramp input, velocity input

■  $k = 2$ : acceleration input

## Steady-State Error by the Final-Value Theorem:

$$\begin{aligned}
 e_{ss} &= \lim_{t \rightarrow \infty} e(t) \\
 &= \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + G D_{cl}} R \\
 &= \lim_{s \rightarrow 0} s \frac{1}{1 + G D_{cl}} \frac{1}{s^{k+1}}
 \end{aligned}$$

### Type 0:

- If  $GD$  has no pole at the origin (no integrator)
- With a unit-step input:

$$\begin{aligned}
 r(t) &= 1(t) \\
 R(s) &= \frac{1}{s}
 \end{aligned}$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + G D_{cl}} \frac{1}{s} = \frac{1}{1 + G D_{cl}(0)}$$

$$\frac{e_{ss}}{r_{ss}} = \frac{e_{ss}}{1} \Rightarrow e_{ss} = \frac{1}{1 + G D_{cl}(0)}$$

### Position Error Constant:

$$G D_{cl}(0) \triangleq K_p$$

- In general case:

$$GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n} \quad GD_{cl}^0(s) \text{ has no pole at the origin}$$

$$GD_{cl}^0(0) = K_n$$

- $n = 0$ :  $GD_{cl}$  has no integrator
- $n = 1$ :  $GD_{cl}$  has one integrator
- $n = 2$ :  $GD_{cl}$  has two integrators

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{GD_{cl}^0(s)}{s^n}} \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

■ In general case: 
$$e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

■ If  $n > k$ , 
$$e_{ss} = 0$$

■ If  $n < k$ , 
$$e_{ss} \rightarrow \infty$$

■ If  $n = k = 0$ , 
$$e_{ss} = \frac{1}{1 + K_n}$$

■ If  $n = k \neq 0$ , 
$$e_{ss} = \frac{1}{K_n}$$

■ Type 0:

- $n = k = 0$

- a step (position) input

- zero-degree polynomial

- $K_0$ : position constant,  $K_0 = K_p$

- In general case: 
$$e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$$

## Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1 + K_p}$	$\infty$	$\infty$
Type 1	0	$\frac{1}{K_v}$	$\infty$
Type 2	0	0	$\frac{1}{K_a}$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0 \qquad GD_{cl}(s) = \frac{GD_{cl}^0(s)}{s^n}$$

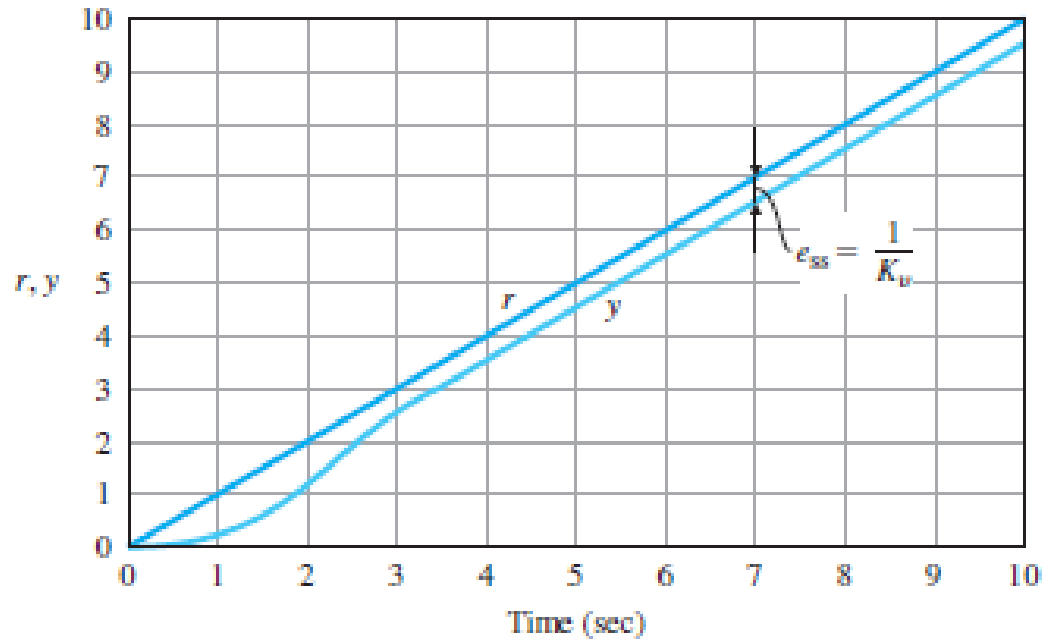
$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

- System Type**: a robust property with respect to parameter changes in the unity feedback structure.

■ In general case:  $e_{ss} = \lim_{s \rightarrow 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}$

● Relationship between ramp response and  $K_v$





## ■ Example 4.1: System Type for Speed Control

$$G(s) = \frac{A}{\tau s + 1}$$

$$D(s) = k_p$$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

$$\Rightarrow GD_{cl}(s) = \frac{k_p A}{\tau s + 1}$$

$\Rightarrow$  no pole at  $s = 0$ ,  $n = 0$

$\Rightarrow$  Type 0,  $K_p = k_p A$

## ■ Example 4.2: System Type using Integral Control

$$G(s) = \frac{A}{\tau s + 1}$$

$$D(s) = k_p + \boxed{\frac{k_I}{s}}$$

$$K_p = \lim_{s \rightarrow 0} GD_{cl}(s), \quad n = 0$$

$$K_v = \lim_{s \rightarrow 0} s GD_{cl}(s), \quad n = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 GD_{cl}(s), \quad n = 2$$

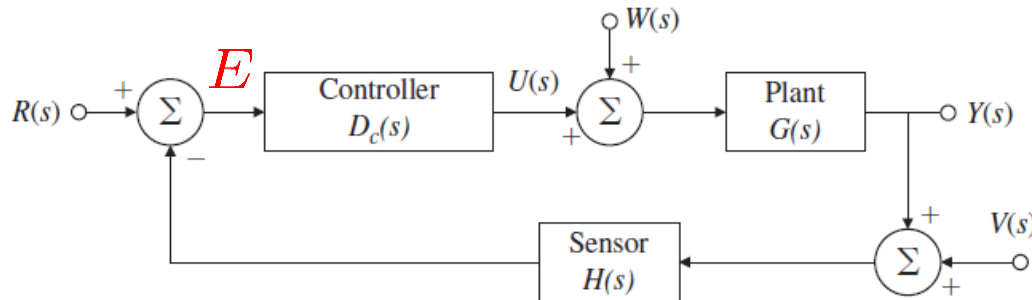
$$\Rightarrow GD_{cl}(s) = \frac{A (k_p s + k_I)}{s (\tau s + 1)}$$

$\Rightarrow$  single pole at  $s = 0$ ,  $n = 1$

$\Rightarrow$  Type 1,  $K_v = k_I A$

- Open-loop system with sensor dynamics.

R = reference, U = control, Y = output, V = sensor noise



$$\begin{aligned} \frac{Y(s)}{R(s)} &= \mathcal{T}(s) \\ &= \frac{G D_c}{1 + G D_c H} \end{aligned}$$

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s) R(s) = R(s) [1 - \mathcal{T}(s)]$$

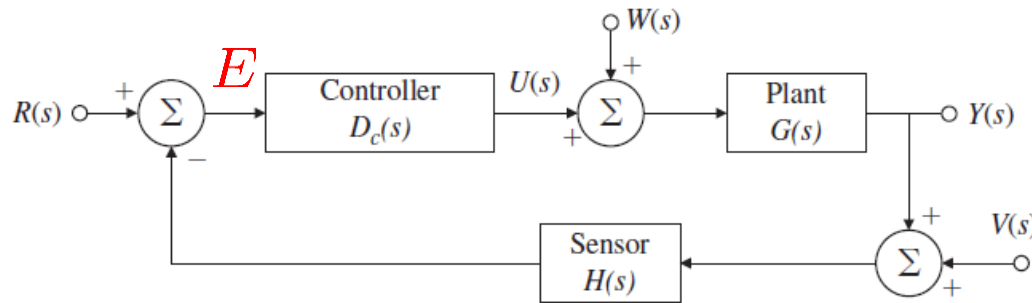
$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s [1 - \mathcal{T}(s)] R(s)$$

- If the reference input a polynomial of degree k:

$$E(s) = \frac{1}{s^{k+1}} [1 - \mathcal{T}(s)]$$

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1 - \mathcal{T}(s)}{s^{k+1}} = \lim_{s \rightarrow 0} \frac{1 - \mathcal{T}(s)}{s^k}$$

## Example 4.3: System Type for Servo w/ Tachometer Feedback



$$G(s) = \frac{1}{s(\tau s + 1)}$$

$$D_c(s) = k_p$$

$$H(s) = 1 + k_t s$$

$$E(s) = R(s) - Y(s) = R(s) - \mathcal{T}(s) R(s)$$

$$= R(s) - \frac{G D_c}{1 + H G D_c} R(s) = \frac{1 + (H - 1)G D_c}{1 + H G D_c} R(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} s R(s) [1 - \mathcal{T}(s)] \quad R(s) = \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} \frac{1 - \mathcal{T}(s)}{s^k} = \lim_{s \rightarrow 0} \frac{1}{s^k} \frac{s(\tau s + 1) + (1 + k_t s - 1)k_p}{s(\tau s + 1) + (1 + k_t s)k_p}$$

$$= 0 \quad k = 0$$

$$= \frac{1 + k_t k_p}{k_p}, \quad k = 1$$

- Transfer function from disturbance input  $W(s)$  to error  $E(s)$

$$\frac{E(s)}{W(s)} = -\frac{Y(s)}{W(s)} = \mathcal{T}_w(s) \quad \text{with } R(s) = 0$$

$$\mathcal{T}_w(s) = s^n \mathcal{T}_{w,0}(s) \quad \text{with } \mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$$

$$y_{ss} = \lim_{s \rightarrow 0} s \mathcal{T}_w(s) \frac{1}{s^{k+1}}$$

$$= \lim_{s \rightarrow 0} \mathcal{T}_{w,0}(s) \frac{s^n}{s^k}$$

- If  $n > k$ ,

$$y_{ss} = 0$$

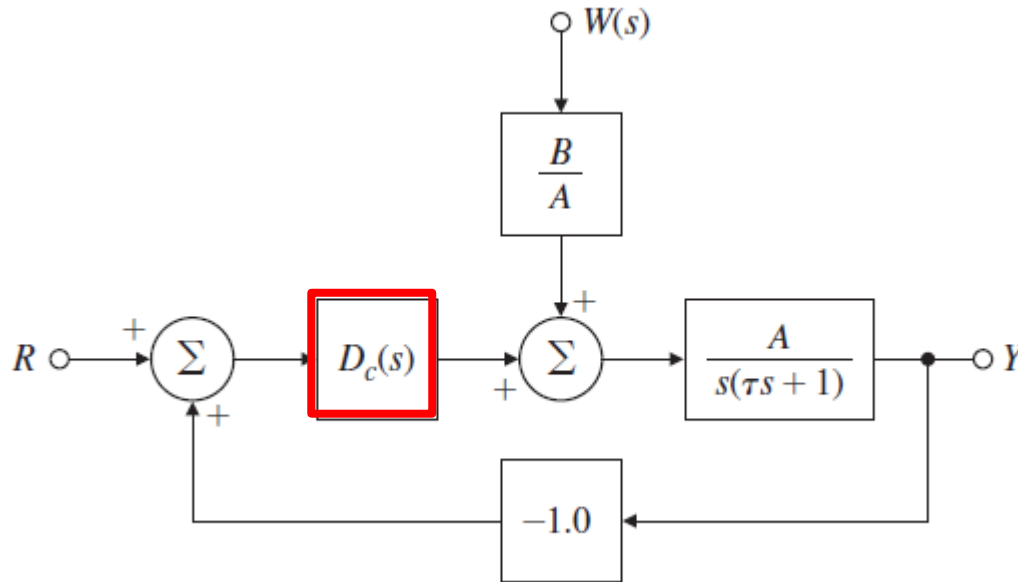
- If  $n < k$ ,

$$y_{ss} \rightarrow \infty$$

- If  $n = k$ ,

$$y_{ss} = \frac{1}{K_{w,n}}$$

- Example 4.4: System Type for DC Motor Position Control



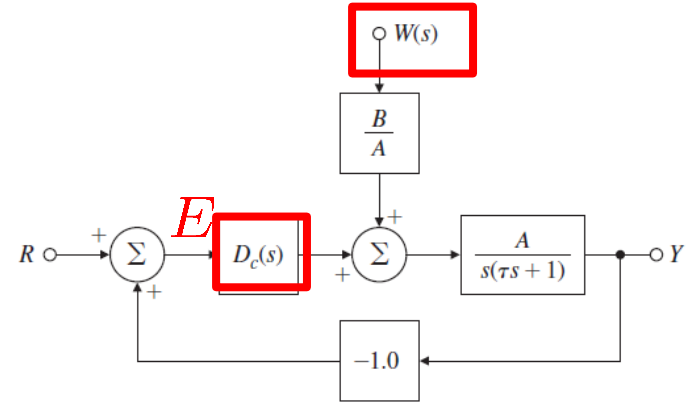
(a)  $D_c(s) = k_P$

(b)  $D_c(s) = k_P + \frac{k_I}{s}$

## Example 4.4: System Type for DC Motor Position Control

(a)  $D_c(s) = k_P$

Transfer function from  $W$  to  $E$  ( $R=0$ )



$$\begin{aligned} \mathcal{T}_w(s) &= -\frac{B}{s(\tau s + 1) + Ak_P} \\ &= s^0 \mathcal{T}_{w,0}(s) \quad n = 0 \end{aligned}$$

$$\mathcal{T}_w(s) = s^n \mathcal{T}_{w,0}(s)$$

$$\text{with } \mathcal{T}_{w,0}(0) = \frac{1}{K_{w,n}}$$

$$K_{w,0} = -\frac{Ak_P}{B}$$

Type 0: Steady-state error to a unit-step torque input is:

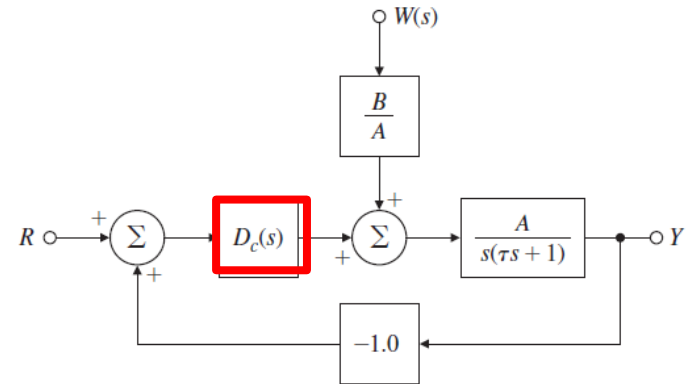
$$e_{ss} = -\frac{B}{Ak_P}$$

$$y_{ss} = \frac{1}{K_{w,n}}$$

## Example 4.4: System Type for DC Motor Position Control

(b)  $D_c(s) = k_P + \frac{k_I}{s}$

Transfer function from  $W$  to  $E$  ( $R=0$ )



$$T_w(s) = - \frac{Bs}{s^2(\tau s + 1) + (k_P s + k_I)A}$$

$$n = 1$$

$$T_w(s) = s^n T_{w,0}(s)$$

$$K_{w,n} = - \frac{Ak_I}{B}$$

$$\text{with } T_{w,0}(0) = \frac{1}{K_{w,n}}$$

Type 1: Steady-state error to a unit-ramp disturbance input is:

$$e_{ss} = - \frac{B}{Ak_I}$$

$$y_{ss} = \frac{1}{K_{w,n}}$$