Spring 2020

控制系統
Control Systems

Unit 36
Stability

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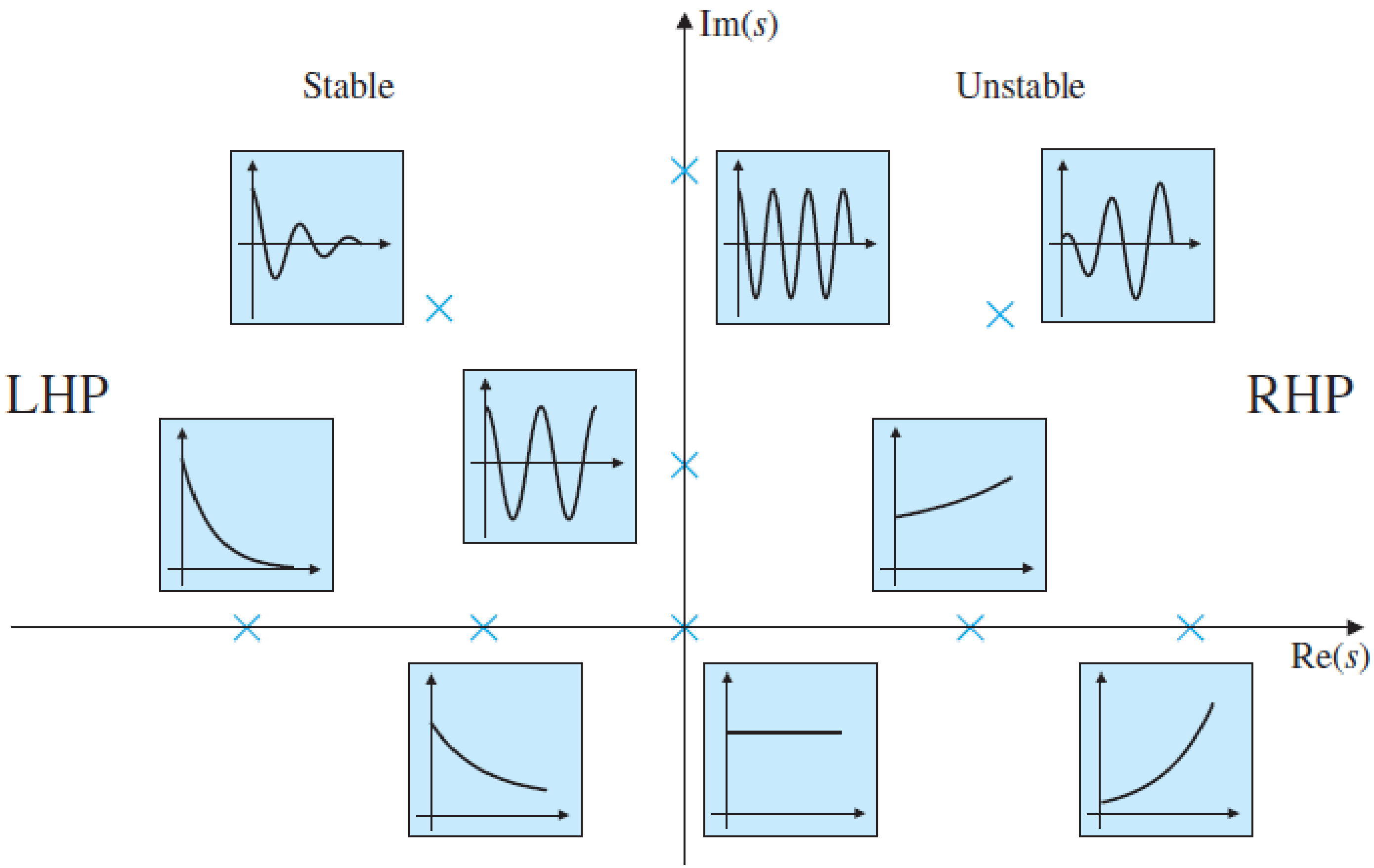
- Stability:
  - An LTI system is said to be stable
    if all the roots of the transfer function
    denominator polynomial have negative real parts
    (that is, they are all in the left-hand s-plane)
    and is unstable otherwise.

- Stable System:
  - A system is stable
    if its initial conditions decay to zero and
    is unstable if they diverge.

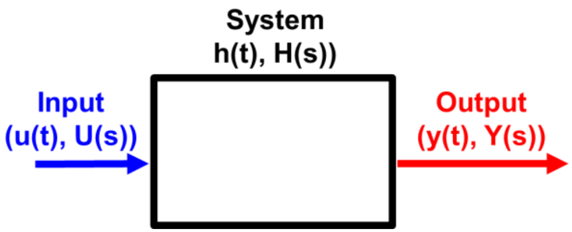
- Time functions associated with points in the s-plane
  (LHP, left half-plane; RHP, right half-plane)



- Bounded Input-Bounded Output Stability (BIBO Stable)
  - A system is said to have BIBO stability

    if every bounded input results in a bounded output

    (regardless of what goes on inside the system).



- If the system has

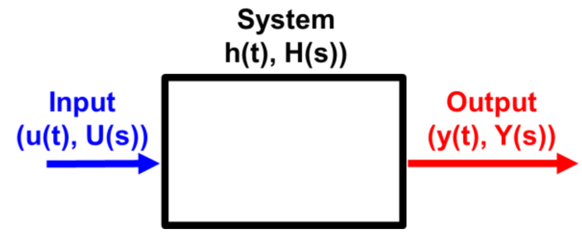
  input u(t), output y(t), and impulse response h(t),

  then

$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$

Bounded Input-Bounded Output Stability

$$y(t) = \int_{-\infty}^{\infty} h(t)\ u(t-\tau)\ d\tau$$



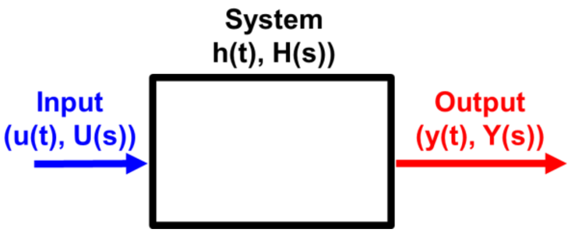
- If input u(t) is bounded $\quad |u(\cdot)| \ \leq \ M \ < \ \infty$
- And output y(t) is bounded by

$$|y| = \left| \int_{-\infty}^{\infty} h\ u\ d\tau \right| \quad \leq \int_{-\infty}^{\infty} |h|\ |u|\ d\tau \quad \leq M \int_{-\infty}^{\infty} |h|\ d\tau$$

- That is, output y(t) is bounded if $\displaystyle\int_{-\infty}^{\infty} |h|\ d\tau$ is bounded.

Bounded Input-Bounded Output Stability

$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$



- On the other hand,

  suppose the integral is not bounded and the input is bounded,
- e.g., $u(t-\tau) = +1$ if $h(\tau) > 0$

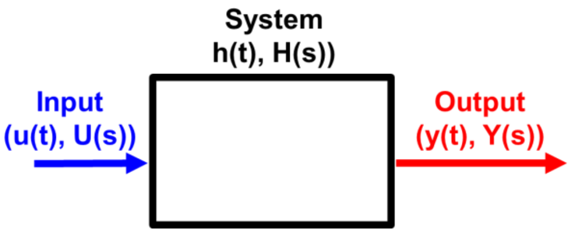
  and $u(t-\tau) = -1$ if $h(\tau) < 0$

$$|y| = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

- The output y(t) is not bounded.

Bounded Input-Bounded Output Stability

$$y(t) = \int_{-\infty}^{\infty} h(t)\, u(t-\tau)\, d\tau$$



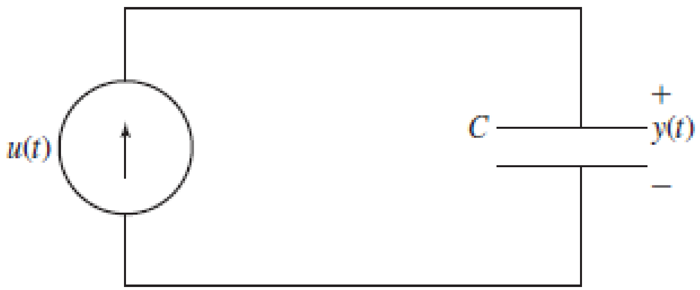
- Mathematical Definition of BIBO Stability
  - The system with impulse response h(t) is BIBO stable

    if and only if

    the integral $$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

▪ Example 3.31: BIBO Stability for a Capacitor

$$\Rightarrow \ h(\tau) \ = \ 1(t)$$



- Capacitor driven by current source

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau \ = \ \int_{-\infty}^{\infty} d\tau \ \rightarrow \ \infty$$

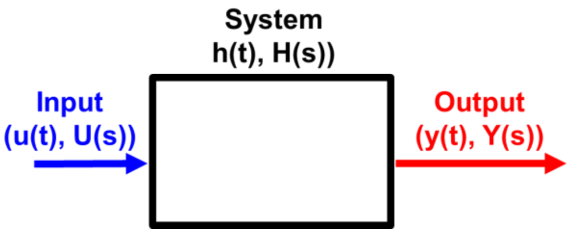
- Consider the LTI whose transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}$$

$$= K \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}, \quad m \leq n$$

$$\Rightarrow y(t) = \sum_{i=1}^{n} K_i \, e^{p_i t}$$

- p_i are the roots of a(s), denominator polynomial
- K_i depend on the initial condition and zero locations



- The system is stable

  if and only if (necessary and sufficient condition)

  every term goes to zero as time goes to infinity

$$\Rightarrow \quad e^{p_i t} \to 0, \quad \text{as} \quad t \to \infty, \quad \text{for all } p_i$$

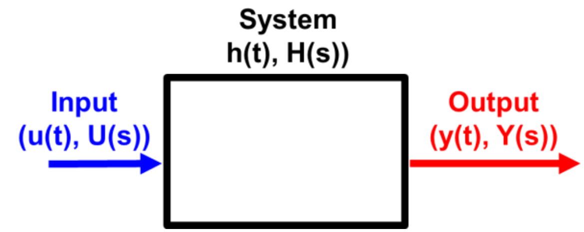
- This will happen

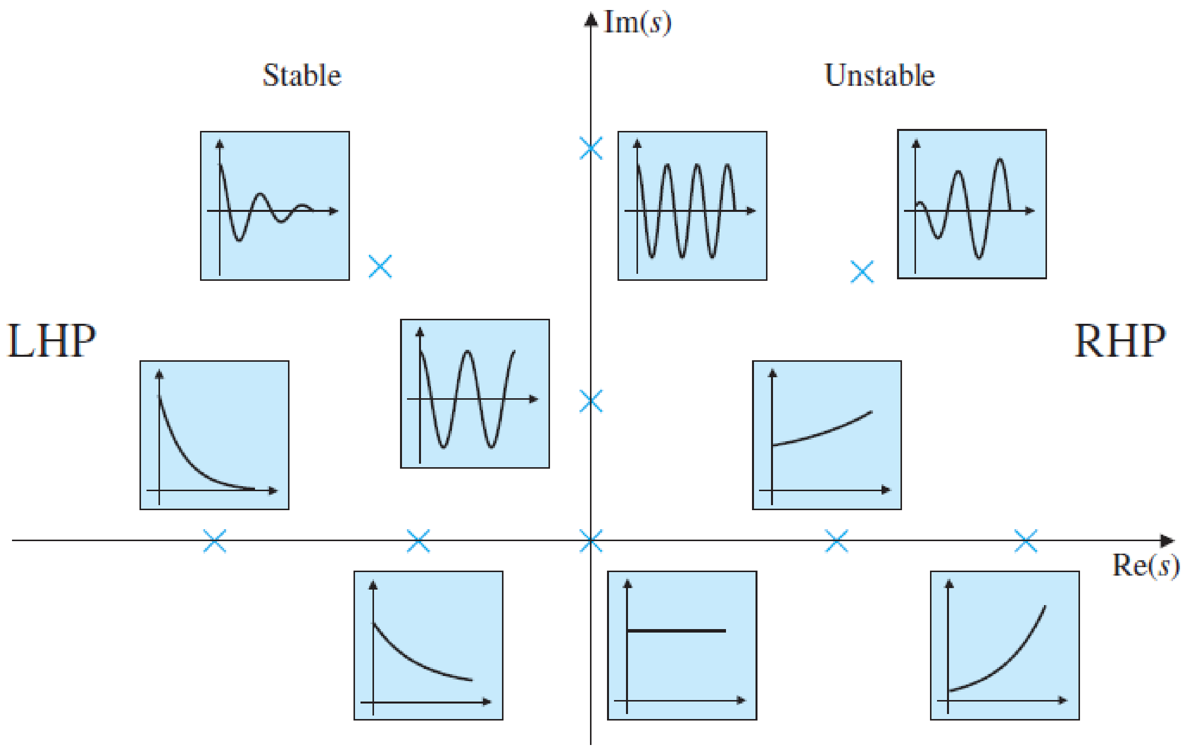
  if all the poles of the system are strictly in the LHP:

$$\Rightarrow \quad \text{Re}\{ \, p_i \, \} \, < \, 0$$

- This is called internal stability

Stability of LTI Systems





stable

neutrally stable

unstable

- Consider the characteristic equation of an nth-order system:

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- A necessary (but not sufficient) condition

  for stability of the system is that

  all of the roots have negative real parts

  which in turn requires that

  all the $\{ a_i \}$ be positive.

- Equivalent test were independently proposed

  by Routh in 1874 and Hurwitz in 1895.

- Routh showed that

  a system is stable if and only if

  all the elements in the first column of the Routh array

  are positive.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- Routh array:

$$\begin{array}{llll}
\text{Row } n & s^n & : & 1 \quad a_2 \quad a_4 \quad \cdots \\
\text{Row } n-1 & s^{n-1} & : & a_1 \quad a_3 \quad a_5 \quad \cdots \\
\text{Row } n-2 & s^{n-2} & : & b_1 \quad b_2 \quad b_3 \quad \cdots \\
\text{Row } n-3 & s^{n-3} & : & c_1 \quad c_2 \quad c_3 \quad \cdots
\end{array}$$

$$b_1 = -\frac{\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1}$$

$$b_2 = -\frac{\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$b_3 = -\frac{\det \begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1}$$

- Routh showed that

  a system is stable if and only if

  all the elements in the first column of the Routh array

  are positive.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- Routh array:

Row n	s^n	:	1	a_2	a_4	$\cdots$
Row $n-1$	s^{n-1}	:	a_1	a_3	a_5	$\cdots$
Row $n-2$	s^{n-2}	:	b_1	b_2	b_3	$\cdots$
Row $n-3$	s^{n-3}	:	c_1	c_2	c_3	$\cdots$

$$c_1 = -\frac{\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1}$$

$$c_2 = -\frac{\det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

$$c_3 = -\frac{\det \begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1}$$

- Routh showed that

  a system is stable if and only if

  all the elements in the first column of the Routh array

  are positive.

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

- Routh array:

$$\begin{array}{llcllll}
\text{Row } n & s^n & : & 1 & a_2 & a_4 & \cdots \\
\text{Row } n-1 & s^{n-1} & : & a_1 & a_3 & a_5 & \cdots \\
\text{Row } n-2 & s^{n-2} & : & b_1 & b_2 & b_3 & \cdots \\
\text{Row } n-3 & s^{n-3} & : & c_1 & c_2 & c_3 & \cdots \\
\text{Row } \vdots & \vdots & : & & & & \\
\text{Row } 2 & s^2 & : & * & * & & \\
\text{Row } 1 & s & : & * & & & \\
\text{Row } 0 & s^0 & : & * & & &
\end{array}$$

- Example 3.32: Routh's Test

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{lllll}
s^6 : & 1 & 3 & 1 & 4 \\
s^5 : & 4 & 2 & 4 & 0
\end{array}$$

$$s^4 : \quad -\frac{1 \cdot 2 - 4 \cdot 3}{4} = \frac{5}{2} \qquad -\frac{4 \cdot 1 - 4 \cdot 1}{4} = 0 \qquad -\frac{1 \cdot 0 - 4 \cdot 4}{4} = 4$$

$$s^3 : \quad -\frac{4 \cdot 0 - 5/2 \cdot 2}{5/2} = 2 \qquad -\frac{4 \cdot 4 - 5/2 \cdot 4}{5/2} = -\frac{12}{5}$$

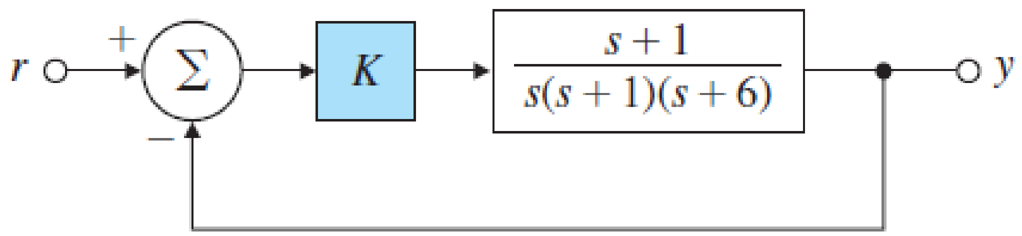
$$s^2 : \quad -\frac{5/2 \cdot (-12/5) - 2 \cdot 0}{2} = 3 \qquad -\frac{5/2 \cdot 0 - 2 \cdot 4}{2} = 4$$

$$s : \quad -\frac{5/2 \cdot (-12/5) - 2 \cdot 0}{2} = -\frac{76}{15} \qquad 0$$

$$s^0 : \quad -\frac{0 - (-76/15) \cdot 4}{(-76/15)} = 4$$

Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability



The characteristic equation for the system:

$$1 + K \frac{s+1}{s(s-1)(s+6)} = 0 \qquad s^3 + 5\,s^2 + (K-6)\,s + K = 0$$

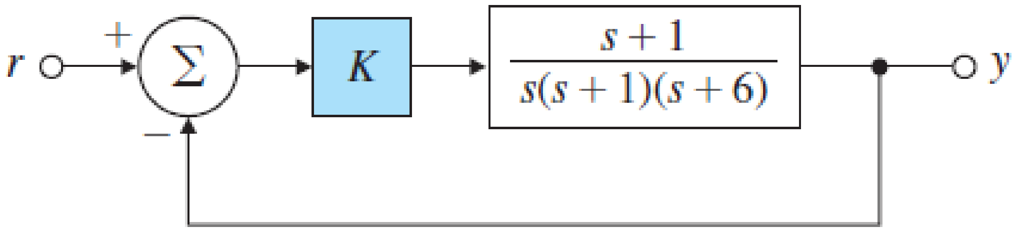
$$\begin{array}{lcll} s^3 & : & 1 & K-6 \\ s^2 & : & 5 & K \end{array}$$

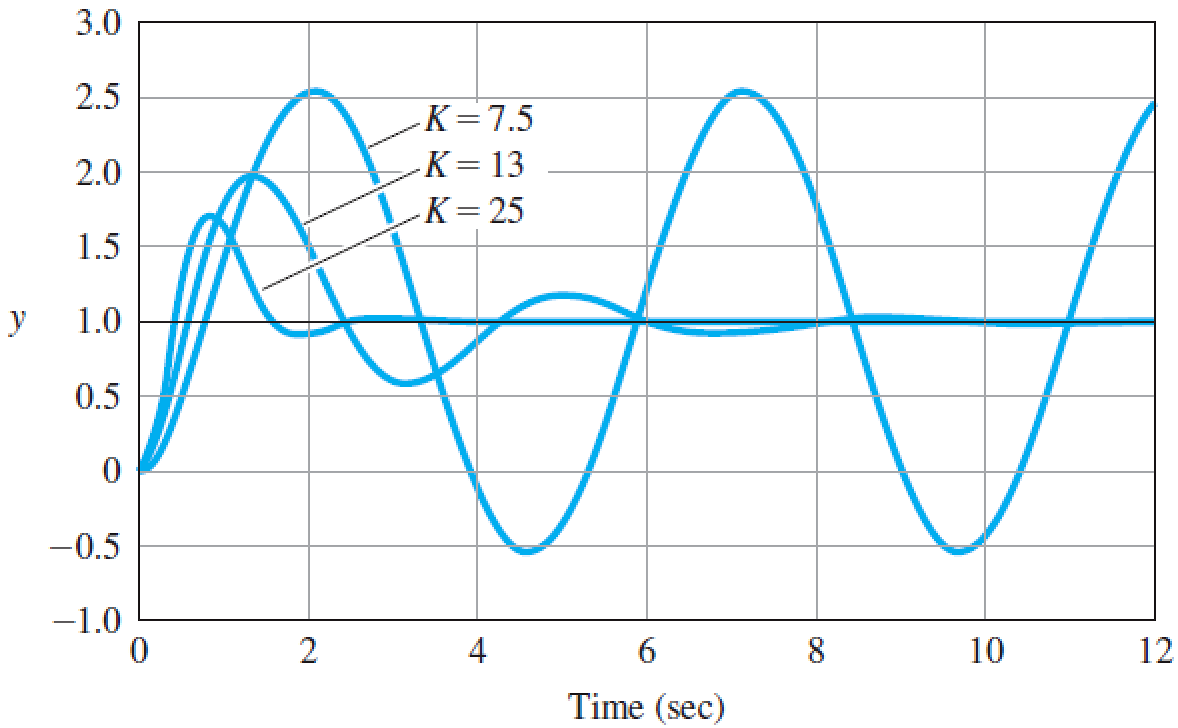
$$\begin{array}{lcl} s & : & (4K-30)/5 \\ s^0 & : & K \end{array} \quad \Rightarrow \quad \frac{(4K-30)}{5} > 0 \quad \Rightarrow \quad K > 7.5$$

$$\Rightarrow \quad K > 0$$

Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability

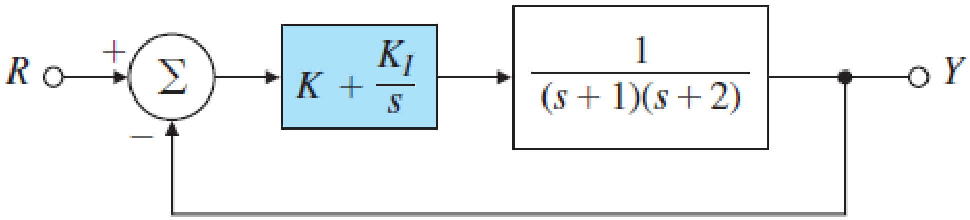




- Transient responses for the system

Example 3.34: Stability versus Two Parameter Ranges

- System with proportional-integral (PI) control



The characteristic equation for the system:

$$1 + \left(K + \frac{K_I}{s}\right) \frac{1}{(s+1)(s+2)} = 0$$

$$s^3 + 3s^2 + (2+K)s + K_I = 0$$

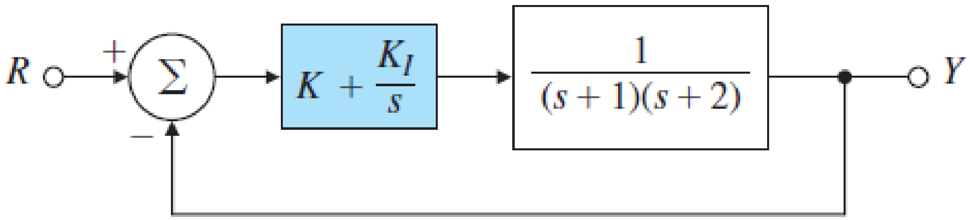
$$\begin{array}{lcll}
s^3 & : & 1 & 2+K \\
s^2 & : & 3 & K_I \\
s & : & (6+3K-K_I)/3 & \\
s^0 & : & & K_I
\end{array}$$

$$\Rightarrow \quad K > \frac{1}{3}K_I - 2$$

$$\Rightarrow \quad K_I > 0$$

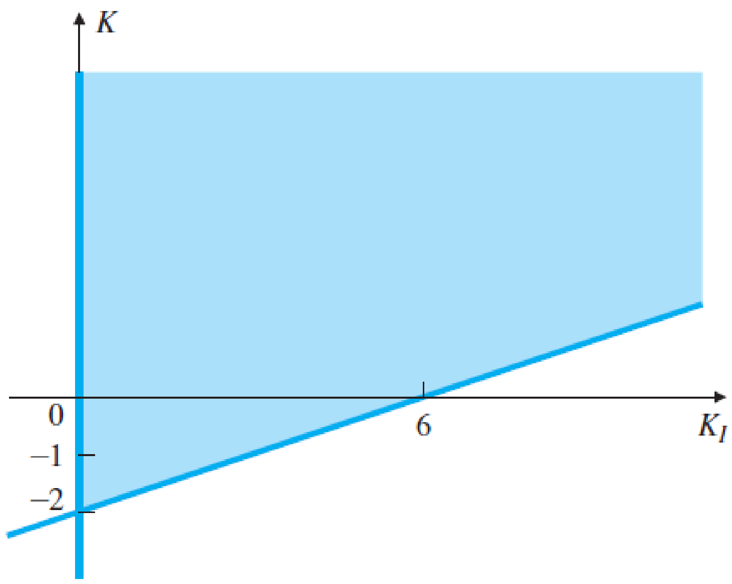
Example 3.34: Stability versus Two Parameter Ranges

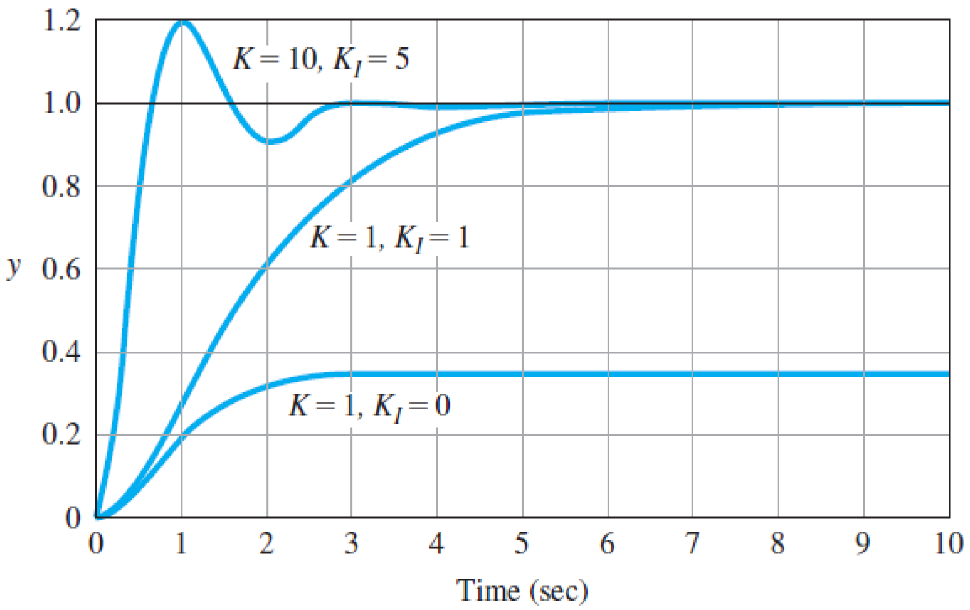
- System with proportional-integral (PI) control



$$\Rightarrow \quad K > \frac{1}{3}K_I - 2$$

$$\Rightarrow \quad K_I > 0$$





- Allowable region for stability
- Transient response for the system