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曲線流的相關研討
A Survey of Curve Shortening Flow

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A Survey of Curve Shortening Flow

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中文摘要

本論文討論二維黎曼流形中的封閉嵌入曲線在曲線流之下的性質。

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A Survey of Curve Shortening Flow

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1 Introduction

There are many ways to flow a submanifold in an ambient manifold. Perhaps the most well-known one is mean curvature flow, the L^2-gradient flow of volume functional. In the one-dimentional case, we would like to know how a smooth immersed curve $C(s,0)$ in a Riemannian surface F, deforms by moving in the direction of its curvature vector field, i.e.

$$\frac{\partial}{\partial t}C(s,t) = k(s,t)N(s,t).$$

We specify its name to be Curve Shortening Flow(CSF). There are several nice properties of the one-dimensional case, for example, the enclosed area of a closed curve in $\mathbb{R}^2$ is decreasing with a constant rate 2π (however this is not true when the ambient surface is not planar.) So it is easy to see that the curve must encounter a singularity at finite time. Let us denote this area-vanishing time by $\widetilde{T}$. On the contrast, when we flow a closed surface in $\mathbb{R}^3$, the enclosed volume of a surface in $\mathbb{R}^3$ may not decrease for all time. For instance, the volume of a bowling ball which has many sharp finger holes will increase at first when the surface evolves under the mean curvature flow.

In this survey, we describe the results and methods derived by M. Gage, R. S. Hamilton, and M. Grayson in the 1980's. In [5],[6], M. Gage proved that when CSF keeps well-defined, the isoperimetric ratio of a convex plane curve is decreasing and the normalized curve converges to a unit circle in the sense that r_{out}/r_{in} converges to 1 as time goes to the area-vanishing time $\widetilde{T}$, where r_{out} and r_{in} stand for the radiuses of the smallest circumscribed circle and the largest inscribed circle respectively. In [7], M. Gage and R. S. Hamilton proved that CSF has short time existence in a rather general situation and every embedded plane curve (may be nonconvex) under CSF remain embedded before the curvature blows up. Moreover, if the initial curve is convex, then the evolution process exists before the area vanishes. They also prove that the curve converges to a round point in C^∞, that is, all higher derivatives of k converge to zero uniformly as $t \to \widetilde{T}$. (In particular, the curve converges in C^2 in the sense that k_{max}/k_{min} converges to 1, where $k_{max}(k_{min})$ is the maximum(minimum) of k on the curve C.) From now on, we use the convention that a strictly convex curve has curvature $k > 0$ with respect to the inward pointing normal vector.

How about a nonconvex plane curve? Does it develop singularities before the enclosed area vanishes? Intuitively speaking, if the curve does not curl intensely, then the concave arcs of it will move outwards such that the minimum of its curvature increases. But what happens to a spiral with a great many turns? Does it loosen speedily before collapsing to its center at $\widetilde{T}$? In [8], M. Grayson proved that if C shrinks to a point as $t \to \widetilde{T}$, then the total absolute curvature of C converges to 2π. In fact, it attains 2π for some $t_0 < \widetilde{T}$, that is, C becomes convex at t_0. He also proved the other situation that C does not shrink to a point never occurs, hence obtained the main theorem in [8] which stated that a nonconvex embedded plane curve becomes convex without developing singularities. Therefore, under CSF, all embedded plane curves have the same asymptotic behavior because they all vary to be convex before the area-vanishing time $\widetilde{T}$.

When we consider CSF of an embedded curve in a Riemannian surface, because a curve cannot leave its convex hull, it is natural to request the convex hull of every compact subset of the surface to be compact. This condition is called "convex at infinity." Under such condition, the reasonable limiting shape of a curve under CSF we may guess is a point or a geodesic. In [9], M. Grayson proved that if the curve does not shrink to a point at finite time, then the curve exists for all time and its geodesic curvature converges to zero in the C^∞-norm. The main idea of the proof is analogous to the $\mathbb{R}^2$ case.

We do not talk about neither the case of immersed curves nor the higher codimensional case, which are discussed in [1], [2], [3], [4], etc. Although there are some remarkable results, the whole picture is not finished yet. So we concentrate on embedded curves in a Riemannian surface in this survey.

2 Notation and preliminaries

For a given embedded curve $C(u,0): S^1 \to F$, where F is a Riemannian surface, we would like to find a natural way to decrease its total length. For this purpose, we consider the flow equation $\partial C(u,t)/\partial t = k(u,t)N(u,t)$, and for convenience, we sometimes use the arc-length parameter s to do the computation. Note that the arc-length parameter s for the curve $C(u,t)$ is not only a function of u but also depends on t. We use ∇_s and ∇_t to denote the covariant differentiation in the space and time directions, respectively. Let $v = \sqrt{\langle \partial C/\partial u, \partial C/\partial u\rangle}$ and $T = \partial C/\partial s$, then the length of $C(s,t)$, $L(t)$, is given by $\int_{S^1} v du = \int_0^{L(t)} ds$.

We derive a technical lemma which would be used later.

Lemma 2.1
(a)

$$\frac{\partial v}{\partial t} = -k^2 v, \quad \nabla_t T = \frac{\partial k}{\partial s} N \quad \textit{and} \quad \nabla_t N = -\frac{\partial k}{\partial s} T.$$

(b)

$$\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s + kR(T,N).$$

Proof.
(a)

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\sqrt{\left\langle \frac{\partial C}{\partial u}, \frac{\partial C}{\partial u}\right\rangle} = \frac{1}{v}\left\langle \nabla_u \frac{\partial C}{\partial t}, \frac{\partial C}{\partial u}\right\rangle = \left\langle \nabla_u(kN), \frac{1}{v}\frac{\partial C}{\partial u}\right\rangle \\ &= -\left\langle kN, \nabla_u \frac{\partial C}{\partial s}\right\rangle = -\langle kN, vkN\rangle = -k^2 v, \end{aligned}$$

$$\begin{aligned} \nabla_t T &= \left(\frac{\partial}{\partial t}\frac{1}{v}\right)\frac{\partial C}{\partial u} + \frac{1}{v}\nabla_t \frac{\partial C}{\partial u} = k^2 \frac{\partial C}{\partial s} + \frac{1}{v}\nabla_u \frac{\partial C}{\partial t} = k^2 T + \nabla_s(kN) \\ &= k^2 T + \frac{\partial k}{\partial s} N + k\nabla_s N = \frac{\partial k}{\partial s} N. \end{aligned}$$

And,

$$0 = \frac{\partial}{\partial t}\langle T, N\rangle = \langle \nabla_t T, N\rangle + \langle T, \nabla_t N\rangle$$

implies the third equation.
(b)

$$\begin{aligned}\nabla_t\nabla_s &= \nabla_t(\frac{1}{v}\nabla_u) = (\frac{\partial}{\partial t}\frac{1}{v})\nabla_u + \frac{1}{v}\nabla_t\nabla_u \\ &= \frac{1}{v}k^2\nabla_u + \frac{1}{v}\left(\nabla_u\nabla_t + R(\frac{\partial}{\partial u}, \frac{\partial}{\partial t})\right) \\ &= k^2\nabla_s + \nabla_s\nabla_t + kR(T,N).\end{aligned}$$

□

Lemma 2.2 *The length function $L(t)$ decreases at the rate $\int_0^{L(t)} k^2 ds$, hence the CSF really shortens a curve when its curvature is not identically zero.*

Proof.

$$\frac{\partial L(t)}{\partial t} = \frac{\partial}{\partial t}\int_{S^1} v du = \int_{S^1} -k^2 v du = -\int_0^{L(t)} k^2 ds.$$

□

The fundamental property of CSF in the plane is

Lemma 2.3 *When F is $\mathbb{R}^2$, we have*

$$\frac{\partial A(t)}{\partial t} = -2\pi,$$

where $A(t)$ is the enclosed area of $C(s,t)$.

Proof. By Lemma2.1,

$$\begin{aligned}\frac{\partial A(t)}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{1}{2}\int_{S^1}\langle C, vN\rangle\, du\right) \\ &= \frac{-1}{2}\int_{S^1}\left\langle \frac{\partial C}{\partial t}, vN\right\rangle + \left\langle C, \frac{\partial v}{\partial t}N\right\rangle + \langle C, v\nabla_t N\rangle\, du \\ &= \frac{-1}{2}\int_{S^1} vk - \left\langle C, v^2kN\right\rangle + \left\langle C, -\frac{\partial k}{\partial u}T\right\rangle du \\ &= \frac{-1}{2}\int_{S^1} vk - \left\langle C, v^2kN\right\rangle + \left\langle \frac{\partial}{\partial u}C, kT\right\rangle + \left\langle C, vk^2N\right\rangle du \\ &= -\int_{S^1} vkdu = -2\pi.\end{aligned}$$

□

Hence, the maximal existence time of CSF in $\mathbb{R}^2$ must be finite, namely, the area-vanishing time $\widetilde{T}$. In the case of a general Riemannian surface, the lemma is no longer true since there exist closed geodesics. We denote the maximal existence time as t_∞, which may be finite or infinite.

3 Short time existence

Instead of the method used in [7], which depends on the Nash-Moser inverse function theorem, one can use the classical argument in the theory of partial differential equations to get short time existence of CSF.

Theorem 3.1 *Let F be a smooth Riemannian surface which is convex at infinity. Let $C(\cdot,0) : S^1 \to F$ be a smooth embedded curve. Then $\exists \epsilon > 0$ and $C : S^1 \times [0,\epsilon) \to F$ which satisfies*

$$\frac{\partial}{\partial t} C = kN, \tag{1}$$

where k is the geodesic curvature of C and N is its unit normal vector. Furthermore, each $C(t) \equiv C(\cdot,t)$ is smooth $\forall t \in [0,\epsilon)$.

Proof. From equation (1) and Lemma 2.1, we can derive the evolution equation for the curvature function k.

$$\begin{aligned}
\frac{\partial k}{\partial t} &= \frac{\partial}{\partial t}\langle \nabla_s T, N\rangle \\
&= \langle \nabla_t \nabla_s T, N\rangle + \langle \nabla_s T, \nabla_t N\rangle \\
&= \langle \nabla_s \nabla_t T + k^2 \nabla_s T + kR(T,N)T, N\rangle + \left\langle kN, -\frac{\partial k}{\partial s} T\right\rangle \\
&= \left\langle \nabla_s \left(\frac{\partial k}{\partial s} N\right), N\right\rangle + k^3 + Rk \\
&= \left\langle \frac{\partial^2 k}{\partial s^2} N - k\frac{\partial k}{\partial s} T, N\right\rangle + k^3 + Rk
\end{aligned}$$

Hence we have

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 + Rk, \tag{2}$$

where R is the Gaussian curvature of F restricted to $C(t)$. For arbitrary parameter u, we have

$$\frac{\partial k}{\partial t} = \frac{1}{v^2}\frac{\partial^2 k}{\partial u^2} - \frac{1}{v^3}\frac{\partial v}{\partial u}\frac{\partial k}{\partial u} + k^3 + Rk.$$

Furthermore, a straightforward calculation shows that, for all $n \geq 1$,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial^n k}{\partial s^n} &= \frac{\partial^2}{\partial s^2}\frac{\partial^n k}{\partial s^n} + \left((n+3)k^2 + R\right)\frac{\partial^n k}{\partial s^n} \\ &\quad + \text{previously bounded terms.}\end{aligned}$$

The two equations above are strictly parabolic provided that the curvature and velocity of a smooth closed curve is bounded, hence $C : S^1 \times [0, \epsilon) \to F$ exists and is smooth. □

Suppose T is the maximal time of existence. If $k(s, T)$ is still bounded, then we can extend the solution to $[0,T+\epsilon)$, which contradicts that T is the maximal time of existence. Hence, k must blow up at T. When $F = \mathbb{R}^2$, we would like to show T is exactly the area-vanishing time $\widetilde{T}$ in section 6 while T may be infinity in the noneuclidean case.

4 The cusp theorem and its generalization

Firstly, we talk about CSF in the euclidean plane. If the curvature is bounded before $\widetilde{T}$, then, by Theorem 3.1, CSF exists for all $t < \widetilde{T}$. Instead of estimating the curvature function k directly, in [7] and [8], they consider the following quantity $\overline{k}_\omega(t) = \sup\{b| \ \ |k| > b$ on some subarc of $C(t)$ which has total absolute curvature $\omega\}$. We may call it the *median curvature* of $C(t)$ when $\omega = \pi$ and $C(t)$ is convex.

Now we introduce the cusp theorem in the plane, which states that the curvature cannot become unbounded unless it does so on an arc which turns through an angle of at least π. Notice that the shape of a convex curve is strongly controlled by its subarc which turns more than π. If the subarc has large curvature, then the area enclosed by the curve is small. This is no longer true for a nonconvex curve, because a small subarc may behave arbitrarily even the large complement is under control.

Theorem 4.1 (The Cusp Theorem) *If $\overline{k}_\pi(t)$ is bounded for $t \in [0, T)$, then k is uniformly bounded on the same time interval. Therefore, they blow up at the same time.*

The theorem was proven in [7] for the case of convex curves, while a similar argument which is valid for the nonconvex case is given in [8].

The cusp theorem prevents the curvature from blowing up unless it does so on some arc which turns at least π. We have no idea that whether the arc is just the place at which k blows up or not. Nevertheless, in Theorem 4.2 we can say something about the curvature on an arc which connects two adjacent inflection points and contains the point where the blow up of curvature is realized.

When we choose an (x, y)-coordinate system and consider the angle $\theta(s, t)$ between the tangent vector at $C(s, t)$ and the x-axis, θ is not a globally defined parameter for a nonconvex curve C. Hence, in [8], Grayson focused on the behavior of each arc which is a graph in some local coordinate system. He consider a "local" flow, different from CSF, which keeps the x-coordinate fixed and has the same point-sets as a solution.

Lemma 4.1 *In a chosen Cartesian coordinate system where a piece of $C(t_0)$ is locally a graph over x-axis, the evolution of y is given by*

$$\frac{\partial y}{\partial t} = \frac{y''}{1 + y'^2}, \tag{3}$$

where the x-coordinate is fixed. Moreover, $\theta(x, t) = \tan^{-1}(y'(x, t))$ and $k(x, t)$ also evolved by equations which are strictly parabolic when $|y'|$ is bounded, namely

$$\frac{\partial \theta}{\partial t} = \frac{\theta''}{1 + y'^2}, \tag{4}$$

$$\frac{\partial k}{\partial t} = \frac{k''}{1 + y'^2} + k^3. \tag{5}$$

If $C(0)$ contains a segment with $k \equiv 0$, then by equation (3) and the maximum principle, the segment will disappear instantly as time goes on. In fact, we may assume $C(0)$ have only finitely many inflection points. Notice that θ is a good parameter of each arc which excludes inflection points. Using (4) and the maximum principle again, we gain the following lemma.

Lemma 4.2 *(i) The θ-interval corresponding to an arc connecting two inflection points strictly nests with time.*
(ii) The points of the curve where local maximum and minimum for θ are realized vary continuously with time, i.e. a given inflection point at time t_0 may be traced continuously back in time to an inflection point on the initial curve and forward as well until it disappears.

For abbreviation, we often use the term "inflection (ω-)arc" instead of "arc connecting two adjacent inflection points (which has total absolute curvature ω.)" By an*inflection subarc*, we mean a family of arcs, $\alpha(t) \subset C(t), t_0 \leq t < t_3$, which connects two continuously deformed inflection points. And we may think of the continuously deformed point as a*moving point.* By saying the curvature k blows up on a moving point, we mean that it becomes unbounded as the point deforms continuously.

Lemma 4.2(i) tells us that the total absolute curvature of arcs in an inflection subarc decreases in time, i.e. if an ω_1-arc $\alpha(t_1)$ deforms to be an ω_2-arc $\alpha(t_2)$ in an inflection subarc, then $\omega_1 > \omega_2$. So the quantity k^ω, which is defined as the maximum of k restricted to the ω-arc in an inflection subarc, is a well-defined function of ω. In [8], Grayson showed that if $\pi \geq \omega_1$, then the boundedness of k^{ω_1} implies that k^ω is uniformly bounded on $[t_1, t_3)$. Therefore, if k is bounded on a π-arc $\alpha(t_0)$, then it is still bounded on $\alpha(t)$ for all $t > t_0$.

We say an arc $\alpha(t_0)$ is*nice* if the tangent vectors which point inwards at the endpoints have the same direction. Just like the inflection subarc, we can construct a nice subarc which is a family of nice arcs all having the same inward pointing direction on the moving endpoints.

Theorem 4.2 *Given $C(0)$, there is a constant K, such that if $|k(s_0, t_0)| > K$, then the inflection arc $\alpha(t_0)$ containing $p = C(s_0, t_0)$ has total absolute curvature $\geq \pi$. Therefore, for each endpoint of $\alpha(t_0)$, there exists another point in $\alpha(t_0)$ such that the arc between these two points is nice.*

Proof. Since $C(0)$ has finite number of inflection points, let K be the maximum of the bounds obtained in the following two fashions:

(i)Each inflection subarc which has total absolute curvature $\leq \pi$ at $t = 0$ has bounded curvature as time goes on. (ii)Each time the total absolute curvature of an inflection subarc drops below π, we have a bound on the maximum of its curvature. (Two or more arcs may fuse to make a single arc with total curvature greater than π, this also could happen for only finite number of times.)

Hence, if $k > K$ at some point p, then the inflection arc containing p must have total absolute curvature $\geq \pi$. $\square$

In the noneuclidean case, if k blows up in finite time, Grayson derived a stronger theorem which states that there are finitely many points $\{q_i\}$ such that the moving point where k blows up on it converges to some q_i.

Theorem 4.3 *If k blows up at $t_\infty < \infty$, then there are finitely many points, $q_i \in C(t_\infty)$, such that for any $\epsilon > 0$, there is a $\overline{K} > 0$ such that for any $t < t_\infty$, every point $p(t)$ on $C(t)$ with $|k| > \overline{K}$ lies in an ϵ-neighborhood of some q_i, and the arc $C(t) \cap B_\epsilon(q_i)$ containing $p(t)$ has total curvature at least $\pi - \epsilon$.*

This theorem can be viewed as a noneuclidean version of the cusp theorem, however, it is more powerful because the subarc where k blows up contains the moving point p.

5 The δ-whisker lemma and embeddedness

The δ-whisker lemma is an important tool in the next section. We state and prove the euclidean version of it at first.

Lemma 5.1 (The δ-whisker Lemma: euclidean version) *Given $C(0)$ smooth, then there is a $\delta > 0$ such that if*

(i) $C(t)$ exists for $t < T$,

(ii) $\alpha(t_0)$ is nice for some $t_0 < T$,

(iii) A δ-whisker L is a line segment of length δ, based at a point p on $\alpha(t_0)$, such that L points in the same direction as the inward pointing tangent vectors to the endpoints of $\alpha(t_0)$,

then L is disjoint from $\beta(t_0) = C(t_0) \setminus \alpha(t_0)$. Furthermore, if $\alpha_1(t_0)$ and $\alpha_2(t_0)$ are disjoint nice arcs of $C(t_0)$, then any two line segments, L_1 and L_2 satisfying condition (iii) for $\alpha_1(t_0)$ and $\alpha_2(t_0)$ respectively, are disjoint.

Proof. Let $d(t)$ be the maximal distance which the nice arc $\alpha(t)$ can be translate before it touches $C(t) \setminus \alpha(t)$. By maximum principle and Lemma 4.1, one can prove that $d(t)$ is increasing. Let $\alpha_1(t_0)$ and $\alpha_2(t_0)$ be two disjoint nice arcs of $C(t_0)$ with inward pointing tangent vectors v_1 and v_2, respectively, at endpoints. Then the backwards nice subarcs containing $\alpha_1(t_0)$ and $\alpha_2(t_0)$ are disjoint in the sense that $\alpha_1(t)$ and $\alpha_2(t)$ are disjoint $\forall t < t_0$.

Consider translations of $\alpha_1(t)$ and $\alpha_2(t)$ in the directions v_1 and v_2. Let d_{12} be the maximum distance which the two arcs can be moved before one of them bumps into either $C(t) \setminus \{\alpha_1(t_0) \cup \alpha_2(t_0)\}$ or some translation of the other arc by some amount $\leq d_{12}$. d_{12} is a monotonically increasing function for the reasons that $d(t)$ was. Therefore they are separating, and at any later time, the arcs may be translated a little further. Let δ be the minimum over all choices of disjoint nice $\alpha_1(0)$ and $\alpha_2(0)$ of $d_{12}(0)$. □

Notice that in the euclidean plane, we can compare the behavior of two widely separated arcs by comparing their translates because the plane is flat and translation commutes with evolution. However, when we translate an arc along the leaves of a geodesic foliation $\mathfrak{F}$ on a Riemannian surface, we need to know how the velocity changes under translation. Grayson showed that a local minimum d for the distance between two arcs cannot decrease to zero in finite time if the change of velocity is under control. Using this fact, the δ-whisker lemma could be adapted to the noneuclidean case.

Lemma 5.2 *Suppose that $d(t)$ is a local minimum for the distance measured along the leaves of $\mathfrak{F}$ between two arcs. Suppose that the product λk is bounded from above on either arc, where $\lambda = D_V \log J$, where V is the unit tangent vector field to $\mathfrak{F}$. Then $d(t)$ can decay at most exponentially.*

As a corollary of this lemma, we have

Lemma 5.3 (The Little δ-whisker Lemma) *Choose the orientation of $\mathfrak{F}$ such that $\lambda \geq 0$. If α is an arc which turns through π relative to the leaves of $\mathfrak{F}$ with tangencies at both endpoints and $k < \infty$ on α, then maximum distance α can be translated outwards along the leaves of $\mathfrak{F}$ can decay at most exponentially.*

Now we want to make sure that an embedded curve keeps embedded under CSF if its curvature is bounded. The proof in [7] is valid for all embedded plane curves, but it depends on planar trigonometry too strongly to be generalized to a noneuclidean version. In a Riemannian surface, if the common perpendicular which realizes the minimum distance of two arcs is short enough, then we can extend it to a foliation $\mathfrak{F}$ so that λ has whatever sign we need to show that the local minimum decreases at most exponentially. Any such $\mathfrak{F}$ will keep the same geodesic as local minimum for distance. Similarly, any nearby geodesic can be extended to an $\mathfrak{F}$ which yields a longer local minimum which decreases at most exponentially. Hence, a local minimum for the distance function on $C \times C$ cannot decrease to zero in finite time. The conclusion is that a smooth embedded curve stays embedded when its curvature is bounded.

At last, with a modified definition of nice arc, we state the noneuclidean version of the δ-whisker lemma which bases on Lemma 5.3.

Lemma 5.4 (The δ-whisker Lemma: noneuclidean version) *Given an initial curve, and assume k blows up at $t_\infty < \infty$, there exist a $t_0 < t_\infty$ and a $\delta > 0$ such that for any nice arc β, no point on the curve outside of β intersects the segments of length δ extending from β along the leaves of $\mathfrak{F}$. Furthermore, two such whiskers from disjoint β are disjoint.*

6 Curvature estimates

In [7], Gage and Hamilton proved that $\overline{k}_\pi < L/A$ for a convex plane curve which has length L and enclosed area A, therefore, by cusp theorem and short time existence, CSF exists for all $t < \widetilde{T}$.

For a nonconvex plane curve, the inequality is in general not true. So the argument in [8] is to analyze the behavior of the curve near the singularity. Suppose the curvature blows up at T, by cusp theorem, there exists an π-arc $\alpha(t_0)$, t_0 near T, such that $k > M$ on α for any prescribed M. We may consider a more complicated quantity $\overline{\omega} := \sup\{\omega | \forall\epsilon > 0, \exists t_0 < T$ and ω-arc $\beta(t_0)$ such that $\text{diam}\beta(t_0) < \epsilon$, and either $k > -K$ or $k < K$ on $\beta(t_0)\}$, which characterizes the singularity type of nonconvex curves. The last condition of $\overline{\omega}$ prevents β to be figure-s, hence the large percentage of the total curvature of β

comes from arcs without inflection points, all of whose curvatures have the same sign. By using Theorem 4.2 and δ-whisker lemma, Grayson proved

Lemma 6.1 *If $C(t)$ shrinks to a point as $t \to T$, then there is an $M > -\infty$ such that $k(\cdot, t) > M$ for all $t < T$ and the total absolute curvature of $C(t)$ converges to 2π. Moreover, $\overline{\omega} = 2\pi$.*

Grayson showed that $C(t)$ must shrink to a point as $t \to T$, because $\overline{\omega} \neq 2\pi$ would lead to a contradiction. He also showed that a curve would become convex before time T.

In the rest of this section, we go through the proof in [8] roughly.

Lemma 6.2 $\overline{\omega} \geq \pi$.

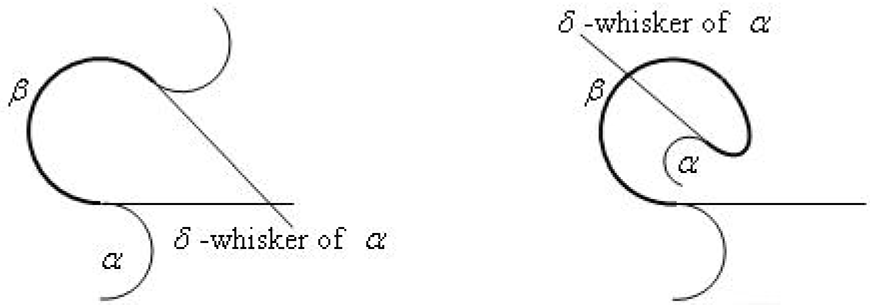
Proof. Since the curvature blows up at T, by Theorem 2.1, for all large number $N > 0$, $\exists \beta(t^*)$ such that $\int_{\beta(t^*)} k = \pi, |k| > N$ on β. Hence diam(β) < half of the length of $\beta \leq \pi/2N$. □

Theorem 6.1 *The case that C does not shrink to a point and $\overline{\omega} > \pi$ never occurs.*

Proof. For any $\epsilon > 0, \exists \beta(t_0)$ such that $\int_{\beta(t_0)} |k| ds = \frac{\overline{\omega}+\pi}{2}$. Construct the outward pointing rays tangent to the endpoints of $\beta(t_0)$. Since either $k > -K$ or $k < K$, if ϵ is small enough then $\beta(t_0)$ can not be decomposed into two arcs whose curvatures have opposite signs and large magnitudes. In fact, an arbitrarily large percentage of the total curvature of $\beta(t_0)$ comes from arcs $\{\beta_i^+(t_0)\}_{i=1,\ldots,n}$ without inflection points, all of whose curvatures have the same sign. Choose ϵ small enough so that $\int_{\beta(t_0) \setminus \cup \beta_i^+(t_0)} |k| < \frac{\overline{\omega}-\pi}{2}$. We may assume the endpoints are either inflection points or lie in $\cup \beta_i^+(t_0)$.

Therefore, either the aforementioned two rays cross, or one of them crosses $\beta(t_0)$. In either event, the crossing must occur within a small neighborhood of the endpoints of $\beta(t_0)$, where small is equal to max$\{\epsilon, \epsilon/(\overline{\omega} - \pi)\}$.

Since $C(t_0)$ is embedded, it must curve very fast to avoid intersection. Choose ϵ sufficiently small, then there exists p near $\beta(t)$ such that $|k(p)| > K$, hence, by Lemma 3.5, $\alpha(t)$, the maximal positive curvature arc containing p, contains a nice arc and $\int_{\alpha(t)} |k| \geq \pi$. Extend $\beta(t)$ until



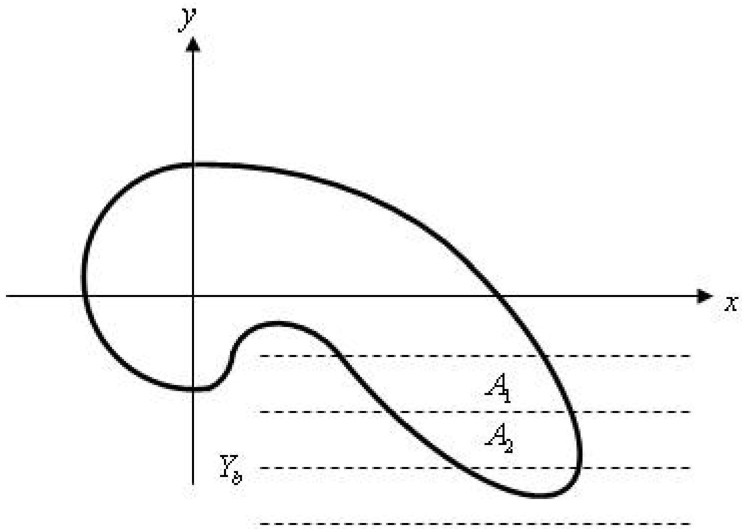
it contacts $\alpha(t)$ at a point of inflection. By δ-whisker lemma, we get a contradiction. □

We can not expect that the same contradiction will occur in the case of $\overline{\omega} = \pi$, because there is no subarc with two intersecting tangent rays at its endpoints. For $\overline{\omega} = \pi$, there is a sequence of arcs $\beta_i(t_i)$ with absolute curvatures greater than K, diameters converging to zero and total absolute curvatures converging to π. A crucial observation is that the two tangent lines at the endpoints of β converge to a single line. We may take this line as x-axis and choose y-axis such that $\beta_i(t_i)$ converges to it from the left. In this Cartesian coordinate system, we argue that $\overline{\omega} = \pi$ is also impossible by using Lemma 4.1 and Lemma 5.1. The proof is much more complicated than the first case, so we just outline the proof below.

Theorem 6.2 *The case that C does not shrink to a point and $\overline{\omega} = \pi$ never occurs, too.*

Proof. Instead of looking at $\beta_i(t_i)$, we consider the inflection subarc $\beta(t)$ which contains $\beta_i(t_i)$. Denote the two arcs to the right of y-axis as β_+ and β_-.

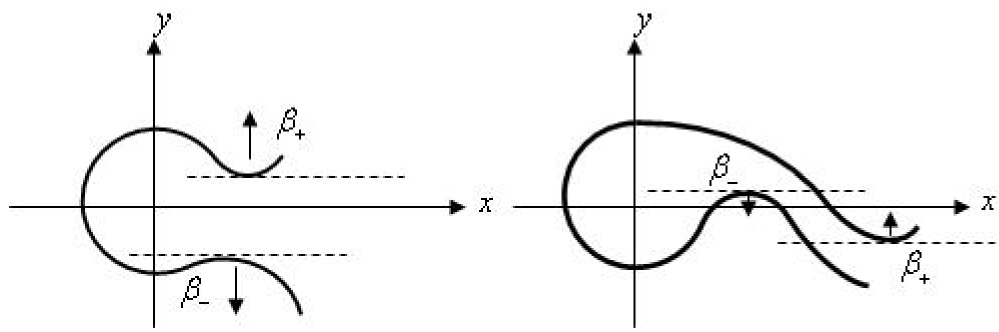
If there is a $\xi > 0$ such that $k > 0$ on $\beta_+ \cap \{(x, y)|0 < x < \xi\}, \forall t \in (t_0, T)$, i.e. there is no moving inflection point converges to the origin along β_+, then $\beta_+(t_0)$ must intersect with the x-axis in $(\xi/2, \xi)$ for some t_0. Hence there must be an inflection point converging to the origin along $\beta_-(t_0)$. Moreover, Grayson proved that the lowest point on the curve is bounded away form the x-axis, that is, the limit Y_b of the height of the lowerest point is less than zero. Consider the areas $A_1(t)$ of the



intersection of the interior of $C(t)$ with the strip $2Y_b/3 \leq y \leq Y_b/3$ and $A_2(t)$ of the intersection of the interior of $C(t)$ with $y \leq 2Y_b/3$, Grayson proved that the horizontal width $h(y,t)$ of $C(t)$ is decreasing in y which implies $A_1(t) \geq h(2Y_b/3, t) \cdot |Y_b/3|$ and $A_2(t) \leq 2 \cdot h(2Y_b/3, t) \cdot |Y_b/3|$, where $t < T$ is large enough such that $C(t)$ lies above the line $y = 4Y_b/3$. Since the area bounded below an arc varies with speed equal to the turning angle of the arc, i.e. $\frac{d}{dt}\int_a^b y dx = \int_{y(a)}^{y(b)} k ds$, one can show that $dA_1/dt \geq -\epsilon$ and $dA_2/dt \leq -\pi + \epsilon$. This is absurd because A_2 will become negative when A_1 goes to zero.

On the other hand, if there is no such ξ, i.e. there are two inflection points converging to the origin along β_+ and β_-, then β_+ and β_- are subarcs with negative curvatures for all t near T. There are two possible situations as depicted in the next page.

The former never occurs because β_+ and β_- always move away from the x-axis while the two inflection points have to converge to the origin. One can show that the latter is also impossible because the horizontal distance between β_+ and β_- will not stay bounded away from zero which contradicts the δ-whisker lemma. □



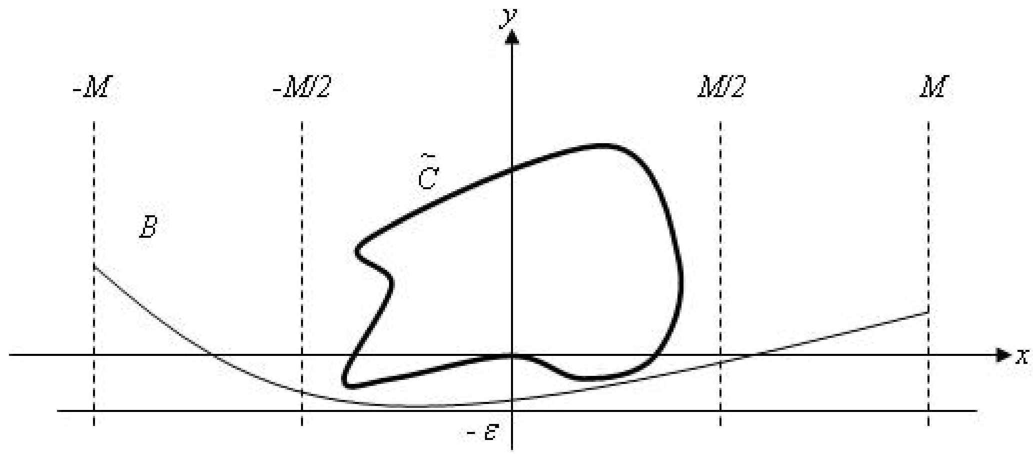
Theorem 6.3 *In the only possible case that C shrinks to a point at T and $\overline{\omega} = 2\pi$, C becomes convex before it shrinks to a point.*

Proof. Let $\widetilde{C}(\tau)$ be the expansion of $C(t)$ which preserves the area enclosed by C and exists for $\tau \in [0, \infty)$, namely,

$$\tau = \ln \frac{A(0)}{A(t)} \qquad \text{and} \qquad \widetilde{C}(\tau) = \sqrt{\frac{d\tau}{dt}} C(t),$$

where $A(t)$ is the enclosed area of $C(t)$ and we may assume $A(0) = 2\pi$ hereafter. Suppose there is an inflection subarc $\beta(t)$ of $C(t)$ with negative curvature for all $t < T$, if there is not, then we are done. By Lemma 6.1, $\int_{\beta(t)} k \to 0$, hence there is a unique direction tangent to $\beta(t)$ for all $t < T$. Choose a Cartesian coordinate system such that the x-axis is parallel to this direction. Since C shrinks to a point, $\widetilde{C}$ must have bounded diameter. Thus, we can choose M so that for all $\epsilon > 0$, $|x| < M/2$ and $y > -\epsilon$ everywhere on some $\widetilde{C}(\tau_\epsilon)$, and $\widetilde{\beta}(\tau_\epsilon)$, the corresponding subarc of $\beta(t)$ under the expansion, has a horizontal tangent at the origin. Consider a "basket" B which is a convex curve satisfying that $y > -\epsilon, |y'| < 1$ everywhere on B and $y > 1$ on B for $x = \pm M$.

Choose ϵ small enough such that, after a half second, y has become positive on the whole B. If we suspend the expansion process, $\widetilde{C}(\tau_\epsilon)$ will shrink to a point after one second. By assumption, $\widetilde{\beta}$ cannot disappear and it must move toward the $-y$ direction, that is, y cannot become positive everywhere on $\widetilde{C}$. This contradicts the fact that $\widetilde{C}$ must lie above B. □



When the ambient manifold F is a Riemannian surface and t_∞ is finite, Grayson used a similar procedure to prove the curve must shrinks to a point. Of course, the δ-whisker lemma of noneuclidean version plays an analogous role as the planar case. Therefore, the curve either shrinks to a point or it exists for infinite time under CSF.

7 Asymptotic behavior

In [7], Gage and Hamilton proved that a convex curve becomes circular when it shrinks to a point under CSF. More precisely, the area-preserving expansion $\widetilde{C}(\tau)$ converges to a circle in C^∞-norm. Also in [9], by using a similar approach, Grayson proved that $C(t)$ converges to a geodesic in C^∞-norm if $C(t)$ does not shrinks to a point in finite time, i.e. $t_\infty = \infty$. To show the C^∞-convergence, it is suffice to prove that $\frac{\partial}{\partial t} \int k^{(n)} \leq -M \int k^{(n)}$, i.e. $\|k^{(n)}\|_\infty$ decays exponentially, where $k^{(n)}$ denotes the n-th derivative of k with respect to the space variable.

The main idea is using the Sobolev inequality and the Peter-Paul inequality $ab \leq \epsilon a^2 + b^2/4\epsilon$. The former tells that $\|f\|_\infty$ is bounded if $\|f\|_2$ and $\|f'\|_2$ are bounded. To show how the latter inequality performs, we give an estimate of $\|\widetilde{k}''\|_2$, which can be found in [7], where $\widetilde{k}$ is the curvature of the area-preserving expansion $\widetilde{C}(\tau)$ of $C(t)$ in the plane. The evolution equation of $\widetilde{k}$ is given by $\frac{\partial}{\partial \tau}\widetilde{k} = \widetilde{k}^2\widetilde{k}'' + \widetilde{k}^3 - \widetilde{k}$, where $'$ denotes $\frac{\partial}{\partial\theta}$. We consider

$$\frac{\partial}{\partial\tau}\int(\widetilde{k}'')^2 = \int -2(\widetilde{k}'')^2 - 2(\widetilde{k}\widetilde{k}''')^2 - 4\widetilde{k}\widetilde{k}'\widetilde{k}''\widetilde{k}''' - 6\widetilde{k}^2\widetilde{k}'\widetilde{k}'''.$$

It is easy to see that there appears to be higher order terms and we use the Peter-Paul inequality to synthesize them,

$$\begin{aligned}\frac{\partial}{\partial\tau}\int(\widetilde{k}'')^2 \quad &\leq \quad \int -2(\widetilde{k}'')^2 - 2(\widetilde{k}\widetilde{k}''')^2 \\ &\quad +4\left(\epsilon(-\widetilde{k}\widetilde{k}''')^2 + \frac{1}{4\epsilon}(\widetilde{k}'\widetilde{k}'')^2\right) + 6\left(\epsilon(-\widetilde{k}\widetilde{k}''')^2 + \frac{1}{4\epsilon}(\widetilde{k}\widetilde{k}')^2\right).\end{aligned}$$

Choose ϵ small to get

$$\frac{\partial}{\partial\tau}\int(\widetilde{k}'')^2 \leq \int -2\alpha\left((\widetilde{k}'')^2 + (\widetilde{k}\widetilde{k}''')^2\right) + \frac{1}{\epsilon}(\widetilde{k}'\widetilde{k}'')^2 + \frac{3}{2\epsilon}(\widetilde{k}\widetilde{k}')^2,$$

where $0 < \alpha < 1$. Moreover, if $\|k'\|_\infty$ is small enough, $\widetilde{k} \to 1$ and $\int(\widetilde{k}'')^2 \leq \int(\widetilde{k}''')^2$, then we can prove that

$$\frac{\partial}{\partial\tau}\int(\widetilde{k}'')^2 \leq -4\alpha\int(\widetilde{k}'')^2 + C_1e^{-4\alpha\tau},$$

and hence $\|\widetilde{k}''\|_2 \leq Ce^{-2\alpha\tau}$ for all $0 < \alpha < 1$ we chose. The conclusion is that we can use the Peter-Paul inequality to combine all higher order terms, even more, we can cancel all higher order terms if needed. Notice that a Poincare type inequality is also needed. This approach is useful for proving C^∞-convergence when the evolution is governed by a heat equation.

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