

# Note on Weak Maximum Principle for Parabolic PDE, Part I

Revised on 2008-7-10

## 1 Weak maximum principle (from Ch. 7 of Evans's PDE book)

Let  $U$  be a **bounded domain** (open and connected) in  $\mathbb{R}^n$  and for  $T > 0$ , we set  $U_T = U \times (0, T]$ . Assume  $L$  is a linear operator in **nondivergent form** (for the purpose of maximum principle) given by

$$Lu = a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t}, \quad (x, t) \in U_T = U \times (0, T]. \quad (1)$$

**Assume all coefficients are continuous on  $\bar{U}_T$**  and the equation is **uniformly parabolic** in the sense that there exists a positive constant  $\theta > 0$ , independent of  $(x, t) \in U_T$ , such that

$$a_{ij}(x, t) \xi_i \xi_j \geq \theta |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad (x, t) \in U_T. \quad (2)$$

**Remark 1** For the purpose the weak maximum principle, in most situation, it suffices to assume that

$$a_{ij}(x, t) \xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^n, \quad (x, t) \in U_T \quad (3)$$

which is same as the matrix inequality  $[a_{ij}(x, t)] \geq 0$  for all  $(x, t) \in U_T$ .

**Remark 2** Note that here the domain  $U_T$  is a **cylinder**, not a general space-time domain in  $\mathbb{R}^{n+1}$ .

**Definition 3** The **parabolic boundary** of  $U_T$  is defined as

$$\Gamma_T = \bar{U}_T - U_T = \bar{U} \times [0, T] - U_T. \quad (4)$$

We also call  $U_T$  the **parabolic interior** of  $\bar{U} \times [0, T]$ . The parabolic boundary  $\Gamma_T$  of  $U_T$  consists of the bottom and vertical sides of  $\bar{U} \times [0, T]$ , but not the top part. More precisely, we have

$$\Gamma_T = \{(x, t) : \text{either } x \in \partial U, 0 \leq t \leq T \text{ or } x \in U, t = 0\}. \quad (5)$$

**Definition 4** The space  $C_1^2(U_T)$  is defined as

$$\{u : U_T \rightarrow \mathbb{R} : u, \partial_i u, \partial_{ij} u, \partial_t u \in C(U_T)\}.$$

In particular,  $u, \partial_i u$  ( $1 \leq i \leq n$ ),  $\partial_{ij} u$  ( $1 \leq i, j \leq n$ ),  $\partial_t u$  are **continuous up to the top**  $U \times \{t = T\}$ .

**In most of the theorems below, the sign of  $c(x, t)$  is crucial.** We first have:

**Theorem 5** (weak maximum principle for  $c \equiv 0$ ) Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$  and

$$c \equiv 0 \quad \text{in } U_T. \quad (6)$$

We have:

$$\text{If } Lu \geq 0 \text{ in } U_T, \quad \text{then } \max_{\bar{U}_T} u = \max_{\Gamma_T} u. \quad (7)$$

Also

$$\text{If } Lu \leq 0 \text{ in } U_T, \quad \text{then } \min_{\bar{U}_T} u = \min_{\Gamma_T} u. \quad (8)$$

Here  $\Gamma_T = \bar{U}_T - U_T$  is the **parabolic boundary** of  $U_T$ .

**Remark 6** (be careful) We do not exclude the possibility that the maximum (minimum) can be attained at some point in the interior  $U_T$ .

**Proof.** First assume that  $Lu > 0$  in  $U_T$ . Since  $u \in C^0(\bar{U}_T)$ , there is a point  $(x, t) \in \bar{U}_T$  such that

$$u(x, t) = \max_{\bar{U}_T} u.$$

If  $(x, t) \in U_T$  with  $t < T$ , the necessary relations

$$\frac{\partial u}{\partial t} = 0, \quad a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \leq 0, \quad b_i \frac{\partial u}{\partial x_i} = cu = 0$$

at the point  $(x, t)$  would contradict  $Lu > 0$  in  $U_T$ . If  $(x, t) \in U_T$  with  $t = T$ , then replace  $\partial_t u(x, t) = 0$  by  $\partial_t u(x, t) \geq 0$ , we still get a contradiction. Thus in such case (i.e.,  $Lu > 0$  in  $U_T$ ) the maximum point  $(x, t)$  **must lie** on the parabolic boundary of  $U_T$  (and can **not** lie on  $U_T$ ). Therefore

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u, \quad \text{if } Lu > 0 \text{ in } U_T.$$

If  $Lu \geq 0$  in  $U_T$ , introduce

$$v(x, t) = u(x, t) - \varepsilon t, \quad \varepsilon \text{ is a positive constant.}$$

Then  $Lv = Lu + \varepsilon > 0$  in  $U_T$  and so

$$\max_{\bar{U}_T} u = \max_{\bar{U}_T} (v + \varepsilon t) \leq \max_{\bar{U}_T} v + \varepsilon T = \max_{\Gamma_T} v + \varepsilon T \leq \max_{\Gamma_T} u + \varepsilon T.$$

Letting  $\varepsilon \rightarrow 0$ , we are done. □

**Theorem 7** (weak maximum principle for  $c \leq 0$ ) Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$  and

$$c \leq 0 \quad \text{in } U_T \tag{9}$$

We have:

$$\text{If } Lu \geq 0 \text{ in } U_T, \quad \text{then } \max_{\bar{U}_T} u \leq \max_{\Gamma_T} u^+. \tag{10}$$

Also

$$\text{If } Lu \leq 0 \text{ in } U_T, \quad \text{then } \min_{\bar{U}_T} u \geq -\max_{\Gamma_T} u^-. \tag{11}$$

Here  $u^+ = \max\{u, 0\} \geq 0$ ,  $u^- = -\min\{u, 0\} \geq 0$ .

**Remark 8** We can **not** replace (10) by  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u^+$  or  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$ . To see this, let  $a_{ij} = \delta_{ij}$  for all  $i, j$ ,  $b_i = 0$  for all  $i$ ,  $c = -1$  and  $u(x, t) = -x$ ,  $x \in U = (1, 2)$ , then it is impossible to have  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u^+$ . For the second case, let  $a_{ij} = \delta_{ij}$ ,  $b_i = 0$ ,  $c = -2$ ,  $u(x, t) = -e^{-t}(x^2 + 2)$ ,  $x \in U = (-1, 1)$ . Then

$$\begin{aligned} & a_{ij} \partial_{ij} u + b_i \partial_i u + cu - \partial_t u \\ &= -2e^{-t} + 2e^{-t}(x^2 + 2) - [e^{-t}(x^2 + 2)] = e^{-t}(x^2) \geq 0 \quad \text{in } U \end{aligned}$$

but we do not have  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$ .

**Remark 9 (important)** (10) implies that if  $\max_{\bar{U}_T} u > 0$ , then  $u$  must attain the positive maximum on the parabolic boundary. If  $\max_{\bar{U}_T} u = 0$ , then  $u$  can also attain 0 on the parabolic boundary. This is due to the strong maximum principle (see Friedman's book).

**Remark 10** Compare with the elliptic case (see Gilbarg & Trudinger, p. 32-33): Let  $\Omega$  be a bounded domain and assume  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ , then

- (1).  $Lu \geq 0$  in  $\Omega$  with  $c(x) \equiv 0$  in  $\Omega$ , then  $\sup_{\Omega} u = \sup_{\partial\Omega} u$ .
- (2).  $Lu \geq 0$  in  $\Omega$  with  $c(x) \leq 0$  in  $\Omega$ , then  $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$ . (12)
- (3).  $Lu \equiv 0$  in  $\Omega$  with  $c(x) \leq 0$ , in  $\Omega$ , then  $\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$ .

Here

$$Lu \equiv a_{ij}(x) D_{ij}u + b_i(x) D_i u + c(x) u \quad (13)$$

and  $a_{ij}(x)$ ,  $b_i(x)$ ,  $c(x)$  are in  $C^0(\bar{\Omega})$ .

**Proof.** First assume  $Lu > 0$  in  $U_T$ . If over  $\bar{U}_T$ ,  $u$  attains a positive maximum at  $(x_0, t_0) \in U_T$ , then  $Lu \leq 0$  at  $(x_0, t_0)$ , which is a contradiction. Hence if  $u$  has a **positive** maximum at  $(x_0, t_0) \in \bar{U}_T$ ,  $(x_0, t_0)$  **must be** on the **parabolic boundary**  $\Gamma_T$ . In such case, we have  $\max_{\bar{U}_T} u = \max_{\Gamma_T} u^+$ . On the other hand if  $u$  does not have a positive maximum on  $\bar{U}_T$ , then we still have (10). Hence in any case we have (10).

Next, if we have only  $Lu \geq 0$  in  $U_T$ , then consider  $u^\varepsilon(x, t) = u(x, t) - \varepsilon t$ . Since  $c \leq 0$ , we have

$$Lu^\varepsilon = Lu + (\varepsilon - c\varepsilon t) > 0 \quad \text{in } U_T \quad (14)$$

and so

$$\max_{\bar{U}_T} (u - \varepsilon t) \leq \max_{\Gamma_T} (u - \varepsilon t)^+.$$

Letting  $\varepsilon \rightarrow 0$ , we can get (10). (One can also argue that if over  $\bar{U}_T$ ,  $u$  attains a positive maximum at some point in  $U_T$  (with  $\max_{\bar{U}_T} u > \max_{\Gamma_T} u^+$ ), then for  $\varepsilon > 0$  small enough,  $u^\varepsilon$  also attains a positive maximum (over  $\bar{U}_T$ ) at some point in  $U_T$ , which gives a contradiction by (14)).

For (11), we consider  $w = -u$ , then  $Lw \geq 0$  in  $U_T$ ,  $w^+ = u^-$ , and so

$$-\min_{\bar{U}_T} u = \max_{\bar{U}_T} w \leq \max_{\Gamma_T} w^+ = \max_{\Gamma_T} u^-.$$

□

**Corollary 11 (weak maximum principle for  $c \leq 0$ )** Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$  and

$$c \leq 0 \quad \text{in } U_T. \quad (15)$$

We have:

$$\text{If } Lu = 0 \quad \text{in } U_T, \quad \text{then } \max_{\bar{U}_T} |u| = \max_{\Gamma_T} |u|. \quad (16)$$

**Proof.** Since  $Lu = 0$  in  $U_T$ , we have

$$\max_{\bar{U}_T} u \leq \max_{\Gamma_T} u^+ \leq \max_{\Gamma_T} |u| \quad \text{and} \quad -\min_{\bar{U}_T} u \leq \max_{\Gamma_T} u^- \leq \max_{\Gamma_T} |u|.$$

Hence we have (16). □

**Example 12** Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$  and

$$c(x, t) \leq \varepsilon \quad \text{in } U_T \quad (\varepsilon > 0 \text{ is some constant}).$$

If  $Lu = 0$  in  $U_T$ , then

$$\max_{\bar{U}_T} |u| \leq e^{\varepsilon T} \cdot \max_{\Gamma_T} |u|. \quad (17)$$

To see this, let  $v(x, t) = e^{-\varepsilon t} u(x, t)$ , then  $v$  satisfies

$$\tilde{L}v = 0, \quad \tilde{L} = L - \varepsilon.$$

The  $\tilde{c}$  for  $\tilde{L}$  satisfies  $\tilde{c}(x, t) \leq 0$  in  $U_T$ . Hence

$$e^{-\varepsilon T} \max_{\bar{U}_T} |u| \leq \max_{\bar{U}_T} |v| = \max_{\Gamma_T} |v| \leq \max_{\Gamma_T} |u|.$$

**Example 13** Let  $U$  be a bounded domain in  $\mathbb{R}^n$  and let  $u$  be a smooth solution of

$$\begin{cases} \partial_t u - \Delta u + cu = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

where  $g(x) \in C^0(\bar{U})$  and the coefficient  $c(x, t)$  satisfies

$$c(x, t) \geq \gamma > 0 \quad (\gamma \text{ is a constant}).$$

Then we have the estimate

$$|u(x, t)| \leq Ce^{-\gamma t}, \quad \forall (x, t) \in U \times (0, \infty). \quad (18)$$

To see this we rewrite the equation as

$$Lu = \Delta u - \partial_t u + \tilde{c}u = 0, \quad \tilde{c} = -c \leq -\gamma < 0$$

and look at  $v(x, t) = e^{\gamma t} u(x, t)$ , then  $v$  satisfies

$$\Delta v - \partial_t v + c^*v = 0, \quad \text{where } c^* = \tilde{c} + \gamma \leq 0$$

and by (16) we have for any  $T > 0$  the equality  $\max_{\bar{U}_T} |v| = \max_{\Gamma_T} |v|$ , which implies

$$e^{\gamma T} |u(x, T)| \leq \max_{\bar{U}_T} e^{\gamma t} |u(x, t)| = \max_{\Gamma_T} e^{\gamma t} |u(x, t)| \leq C.$$

Thus (18) follows.

**Example 14** Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$ . If

$$\begin{cases} Lu = f \in C^0(\bar{U}_T) & \text{in } U_T \\ c(x, t) \leq 0 & \text{in } U_T \end{cases} \quad (19)$$

and

$$a_{11}\lambda^2 + b_1\lambda \geq 1 \quad (20)$$

for some constant  $\lambda > 0$ , then

$$\max_{\bar{U}_T} |u| \leq \max_{\Gamma_T} |u| + (e^{\lambda d} - 1) \max_{\bar{U}_T} |f| \quad (21)$$

where  $d$  is the width of  $\bar{U}_T$  in the  $x_1$ -direction (note that in (20) we look at  $a_{11}$  and  $b_1$ ). That is:  $d$  is the smallest positive number such that, for some  $x_1^0$

$$x_1^0 - d \leq x_1 \leq x_1^0 \quad (22)$$

for all  $x_1$  with  $(x_1, x_2, \dots, x_n, t) \in \bar{U}_T$ . To see this, introduce the function

$$h(x) = 1 - e^{\lambda(x_1 - x_1^0)}, \quad h \geq 0 \quad \text{in } \bar{U}_T.$$

We see that by (22)

$$1 - e^{-\lambda d} \geq 1 - e^{\lambda(x_1 - x_1^0)} \geq 0 \quad \text{in } (x, t) = (x_1, \dots, x_n, t) \in \bar{U}_T$$

Then

$$Lh = - (a_{11}\lambda^2 + b_1\lambda) e^{\lambda(x_1 - x_1^0)} + ch \leq -e^{\lambda(x_1 - x_1^0)} \leq -e^{-\lambda d}.$$

Consider the function

$$w(x, t) = e^{\lambda d} \cdot \max_{\bar{U}_T} |f| \cdot h + \max_{\Gamma_T} |u| \pm u.$$

We have

$$Lw = e^{\lambda d} \cdot \max_{\bar{U}_T} |f| \cdot Lh + c \cdot \max_{\Gamma_T} |u| \pm Lu \leq -\max_{\bar{U}_T} |f| \pm f \leq 0 \quad \text{in } U_T$$

and  $w(x, t) \geq 0$  on the parabolic boundary  $\Gamma_T$ . By the weak maximum principle (11), we have  $w(x, t) \geq 0$  in  $U_T$ , which implies (21). Finally if we replace (19) by

$$\begin{cases} Lu = f \in C^0(\bar{U}_T) & \text{in } U_T \\ c(x, t) \leq \varepsilon & \text{in } U_T \end{cases}$$

where  $\varepsilon > 0$  is some number, then let  $v(x, t) = e^{-\varepsilon t} u(x, t)$ . It satisfies

$$\tilde{L}v = e^{-\varepsilon t} f, \quad \tilde{L} = L - \varepsilon$$

where now the  $\tilde{c}$  for  $\tilde{L}$  satisfies  $\tilde{c}(x, t) \leq 0$  in  $U_T$ . Hence

$$e^{-\varepsilon T} \max_{\bar{U}_T} |u| \leq \max_{\bar{U}_T} |v| \leq \max_{\Gamma_T} |v| + (e^{\lambda d} - 1) \max_{\bar{U}_T} |e^{-\varepsilon t} f| \leq \max_{\Gamma_T} |u| + (e^{\lambda d} - 1) \max_{\bar{U}_T} |f|$$

and so

$$\max_{\bar{U}_T} |u| \leq e^{\varepsilon T} \left\{ \max_{\Gamma_T} |u| + (e^{\lambda d} - 1) \max_{\bar{U}_T} |f| \right\}. \quad (23)$$

**Theorem 15** (weak maximum principle *without* sign condition on  $c$ ) Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$  and  $c$  is **bounded above** on  $\bar{U}_T$ . If

$$Lu \leq 0 \quad \text{in } U_T \quad (24)$$

and  $u(x, t) \geq 0$  on  $\Gamma_T$ , then  $u(x, t) \geq 0$  in  $U_T$ .

**Remark 16** There is **NO** sign condition imposed on  $c(x, t)$  here. The reason is: the boundary value is **zero** ! Precisely, let  $v(x, t) = e^{-\lambda t}u(x, t)$ , then if  $u$  satisfies the equation  $Lu = f$ , then  $v$  satisfies the equation

$$\tilde{L}v = e^{-\lambda t}f, \quad \tilde{L} = L - \lambda$$

Since the condition  $u \geq 0$  on parabolic boundary is same as  $v \geq 0$  on parabolic boundary; for  $v$ , one can make its  $\tilde{c}$  satisfy  $\tilde{c} \leq 0$  by choosing sufficiently large  $\lambda > 0$ .

**Proof.**  $Lu \leq 0$  in  $U_T$  is same as

$$a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} \leq 0 \quad \text{in } U_T.$$

Since  $c$  is bounded above, we assume  $c(x, t) < M$  in  $U_T$  for some constant  $M$ . If  $u(x, t) < 0$  at some interior point of  $U_T$ , then we consider the function<sup>1</sup>

$$v(x, t) = u(x, t)e^{-Mt}. \quad (25)$$

It satisfies  $\frac{\partial v}{\partial t} = e^{-Mt} \frac{\partial u}{\partial t} - Mv$  and so

$$a_{ij}(x, t) \frac{\partial^2 v}{\partial x_i \partial x_j} + b_i(x, t) \frac{\partial v}{\partial x_i} + [c(x, t) - M]v - \frac{\partial v}{\partial t} \leq 0 \quad \text{in } U_T \quad (26)$$

and it assumes a minimal negative value at some point  $(x_0, t_0) \in U_T$ . At this point  $(x_0, t_0)$ , we have

$$\frac{\partial v}{\partial t} \leq 0, \quad \frac{\partial v}{\partial x_i} = 0, \quad a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} \geq 0, \quad (c - M)v > 0$$

which contradicts to (26). □

Similarly we have:

**Corollary 17** (weak maximum principle *without* sign condition on  $c$ ) Assume  $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$  and  $c$  is **bounded above** on  $\bar{U}_T$ . If

$$Lu \geq 0 \quad \text{in } U_T \quad (27)$$

and  $u(x, t) \leq 0$  on  $\Gamma_T$ , then  $u(x, t) \leq 0$  in  $U_T$ .

**Proof.** Let  $w = -u$ . Then

$$Lw \leq 0 \quad \text{in } U_T \quad (28)$$

and  $w(x, t) \geq 0$  on  $\Gamma_T$ , hence  $w(x, t) \geq 0$  in  $U_T$ . □

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<sup>1</sup>This kind of trick is for parabolic equation only.

# Note on Weak Maximum Principle for Parabolic PDE, Part II

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## 2 Weak maximum principle (from Ch. 2 of Lieberman's book)

A point in space-time  $\mathbb{R}^{n+1}$  is denoted as  $X = (x, t)$ , where  $x = (x^1, \dots, x^n)$  will always denote a point in  $\mathbb{R}^n$ . Norms on  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$  are given by

$$|x| = \sqrt{(x^1)^2 + \dots + (x^n)^2} \quad \text{and} \quad |X| = \max \left\{ |x|, |t|^{1/2} \right\}, \quad X = (x, t). \quad (29)$$

**Remark 18** Note that  $t \approx d^2$  (distance square) in parabolic theory.

The set

$$Q(X_0, R) = \{X = (x, t) \in \mathbb{R}^{n+1} : |X - X_0| < R, t < t_0\}, \quad X_0 = (x_0, t_0) \quad (30)$$

is called a *parabolic cylinder* in  $\mathbb{R}^{n+1}$  with center at  $X_0$ . We also use

$$B(x_0, R) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

to denote a ball in  $\mathbb{R}^n$ .

For a  $(n + 1)$ -dimensional domain (open and connected)  $\Omega \subset \mathbb{R}^{n+1}$ , its boundary in  $\mathbb{R}^{n+1}$  is denoted as  $\partial\Omega$ .

**Definition 19** The **parabolic boundary**  $\mathcal{P}\Omega$  of  $\Omega$  is defined as the set of all points  $X_0 \in \partial\Omega$  such that for any  $\varepsilon > 0$ , the cylinder  $Q(X_0, \varepsilon)$  contains points not in  $\Omega$ . Note that in the definition  $\Omega$  may not be bounded in  $\mathbb{R}^{n+1}$ .

**Remark 20** In case  $\Omega = D \times (0, T)$  for some bounded domain in  $\mathbb{R}^n$ , then

$$\mathcal{P}\Omega = (\bar{D} \times \{0\}) \cup (\partial D \times [0, T])$$

where  $\bar{D}$  is the closure of  $D$  in  $\mathbb{R}^n$  and  $\partial D$  is the boundary of  $D$  in  $\mathbb{R}^n$ .

Let

$$Lu = a^{ij}(X) D_{ij}u + b^i(X) D_i u + c(X) u - u_t, \quad X \in \Omega \subset \mathbb{R}^{n+1} \quad (31)$$

where we assume the differential operator  $L$  is **weakly parabolic**, i.e.,

$$a^{ij}(X) \xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^n, \quad \forall X \in \Omega. \quad (32)$$

The simplest version of the maximum principle is the following:

**Lemma 21** Assume  $\Omega$  is a **bounded** domain in  $\mathbb{R}^{n+1}$ . If  $u \in C^{2,1}(\bar{\Omega} \setminus \mathcal{P}\Omega) \cap C^0(\bar{\Omega})$  with  $Lu > 0$  in  $\bar{\Omega} \setminus \mathcal{P}\Omega$  and  $u < 0$  on  $\mathcal{P}\Omega$ , then  $u < 0$  in  $\bar{\Omega}$ .

**Proof.** If not, then  $u \geq 0$  somewhere in  $\bar{\Omega}$ . Let

$$t^* = \inf \{t : u(x, t) \geq 0 \text{ for some } X \in \bar{\Omega}\}.$$

We must have  $t^* > \inf \{t : (x, t) \in \bar{\Omega}\}$  since  $u < 0$  on  $\mathcal{P}\Omega$ . On the *compact* set  $\{x \in \mathbb{R}^n : (x, t^*) \in \bar{\Omega}\}$  there exists a point  $X^* = (x^*, t^*) \in \bar{\Omega} \setminus \mathcal{P}\Omega$  such that  $u(X^*) = 0$ . Also  $u(\cdot, t^*)$  attains its **maximum** at  $x^*$ . If  $X^* \in \Omega$ , then we have (note that  $u(X^*) = 0$ )

$$\begin{cases} Du(X^*) = 0, & a^{ij}(X^*) D_{ij}u(X^*) \leq 0, & u_t(X^*) \geq 0 \\ Lu(X^*) = a^{ij}(X^*) D_{ij}u(X^*) - u_t(X^*) \leq 0 \end{cases} \quad (33)$$

which gives a contradiction.

On the other hand, if  $X^* \in \partial\Omega \setminus \mathcal{P}\Omega$ , one can find a small  $\varepsilon > 0$  such that the cylinder  $Q(X^*, \varepsilon) \subset \Omega$ . Since we assume  $u \in C^{2,1}(\bar{\Omega} \setminus \mathcal{P}\Omega) \cap C^0(\bar{\Omega})$  and  $Lu > 0$  in  $\bar{\Omega} \setminus \mathcal{P}\Omega$ , we still have (33).  $\square$

Some oblique derivative boundary conditions also play an important role. We shall state a version of the weak maximum principle with this type of condition.

For  $X_0 \in \mathcal{P}\Omega$ , we say that a vector  $\beta \in \mathbb{R}^{n+1}$  *points into*  $\bar{\Omega}$  if  $\beta^{n+1} \leq 0$  and there is a positive constant  $\varepsilon$  such that  $X_0 + h\beta \in \bar{\Omega}$  if  $0 < h < \varepsilon$ . For such a  $\beta$ , if  $u$  is *continuous* at  $X_0 \in \mathcal{P}\Omega$  and  $k$  is a constant, we say that  $\beta \cdot \partial u(X_0) \leq k$  if

$$\limsup_{h \rightarrow 0^+} \frac{u(X_0 + h\beta) - u(X_0)}{h} \leq k$$

and  $\beta \cdot \partial u(X_0) \geq k$  if

$$\liminf_{h \rightarrow 0^+} \frac{u(X_0 + h\beta) - u(X_0)}{h} \geq k$$

where  $\partial u$  means  $\partial u = (Du, u_t)$  of  $u$ , the full space-time gradient of  $u$ .

**Remark 22** We say that  $\beta \cdot \partial u(X_0) = k$  if we have both  $\beta \cdot \partial u(X_0) \leq k$  and  $\beta \cdot \partial u(X_0) \geq k$ . Note that  $\beta \cdot \partial u(X_0) = k$  if and only if the corresponding directional derivative  $\partial u / \partial \beta$  exists at  $X_0$  and is equal to  $k$ .

We consider a boundary operator of the form on  $\mathcal{P}\Omega$  :

$$Mu := \beta \cdot \partial u + \beta^0 u \quad (34)$$

where  $\beta$  is a *vector field* on  $\mathcal{P}\Omega$  such that  $\beta(X_0)$  *points into*  $\bar{\Omega}$  at each  $X_0 \in \mathcal{P}\Omega$ , and  $\beta^0$  is a scalar function on  $\mathcal{P}\Omega$ . **For the purpose of the maximum principle, we shall always assume that**

$$\beta^0 < 0 \quad \text{on} \quad \mathcal{P}\Omega. \quad (35)$$

and that  $\beta$  points into  $\bar{\Omega}$  on  $\mathcal{P}\Omega$ .

We now have the following version of the weak maximum principle with oblique derivative boundary condition:

**Lemma 23** Assume  $\Omega$  is a **bounded** domain in  $\mathbb{R}^{n+1}$  and there is a positive constant  $k$  such that

$$c \leq k \quad \text{in} \quad \Omega \quad \text{and} \quad \beta^0 < 0 \quad \text{on} \quad \mathcal{P}\Omega. \quad (36)$$

If  $u \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$  with  $Lu \geq 0$  in  $\Omega$  and  $Mu \geq 0$  on  $\mathcal{P}\Omega$ , then  $u \leq 0$  in  $\Omega$ .

**Remark 24** Lemma 23 says that as long as we have upper bound of  $c$  on  $\Omega$ , then we have the weak maximum principle.

**Proof.** Let  $v = e^{-(k+1)t}u \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$ . Then

$$Lv = e^{-(k+1)t}Lu + (k+1)v \geq (k+1)v \quad \text{in } \Omega \quad (37)$$

and

$$Mv = e^{-(k+1)t}Mu - (k+1)\beta^{n+1}v \geq -(k+1)\beta^{n+1}v \quad \text{on } \mathcal{P}\Omega. \quad (38)$$

If  $v$  has a *positive maximum* at some  $X \in \mathcal{P}\Omega$ , then

$$Mv(X) = (\beta \cdot \partial v + \beta^0 v)(X) \leq (\beta^0 v)(X) < 0 \leq -(k+1)\beta^{n+1}(X)v(X)$$

due to  $(\beta \cdot \partial v)(X) \leq 0$ ,  $\beta^0(X) < 0$ , and  $\beta^{n+1}(X) \leq 0$ . We have got a contradiction due to (38).

On the other hand, if  $v$  has a *positive maximum* at some  $X \in \Omega$ , then

$$\begin{cases} Dv(X) = 0, & a^{ij}(X)D_{ij}v(X) \leq 0, & v_t(X) = 0 \\ Lv(X) \leq c(X)v(X) \leq kv(X) < (k+1)v(X). \end{cases}$$

Again, we got a contradiction due to (37).

From above, if  $v$  has a *positive maximum* on  $\bar{\Omega}$  (note that  $v \in C^0(\bar{\Omega})$ ), it must occur on  $\partial\Omega \setminus \mathcal{P}\Omega$ , say at  $X_0 = (x_0, t_0) \in \partial\Omega \setminus \mathcal{P}\Omega$ . We shall then use *approximation argument*. Set (assume  $i \in \mathbb{N}$  is large enough)

$$t_i = t_0 - \frac{1}{i}, \quad \Omega_i = \{X \in \Omega : t < t_i\} \neq \emptyset, \quad M_i = \sup_{\Omega_i} v \left( = \sup_{\bar{\Omega}_i} v \right) > 0.$$

Also choose  $X_i = (x_i, t_i) \in \bar{\Omega}_i$  so that  $v(X_i) = M_i > 0$ . Since  $\Omega_i \subset \Omega$  and  $\mathcal{P}\Omega_i \subset \mathcal{P}\Omega$ , we must have  $X_i \notin \Omega_i \cup \mathcal{P}\Omega_i$ .

As  $i \rightarrow \infty$ , we have  $v(X_i) \rightarrow v(X_0)$ , so there is a convergent subsequence  $X_{i_j}$  so that  $v(X_{i_j}) \rightarrow v(X_0)$  as  $j \rightarrow \infty$ . Let  $X^* = \lim_{j \rightarrow \infty} X_{i_j}$ . We have  $v(X^*) = v(X_0)$ , so  $X^* \in \partial\Omega \setminus \mathcal{P}\Omega$ . But then there is a positive  $\varepsilon > 0$  such that the cylinder  $Q(X^*, \varepsilon) \subset \Omega$  and hence there is an index  $j$  such that  $X_{i_j} \in Q(X^*, \varepsilon) \subset \Omega$  with  $v(X_{i_j}) = M_{i_j} = \sup_{\bar{\Omega}_{i_j}} v > 0$ . We now have

$$v_t(X_{i_j}) \geq 0, \quad Dv(X_{i_j}) = 0, \quad D^2v(X_{i_j}) \leq 0$$

and

$$(cv)(X_{i_j}) \leq kv(X_{i_j})$$

which gives  $(Lv)(X_{i_j}) \leq kv(X_{i_j})$ , contradicting to (37) again since  $X_{i_j} \in \Omega$ . The proof is done.  $\square$

Now we can get an upper bound of  $u$  on  $\Omega$  from a lower bound of  $Mu$  on  $\mathcal{P}\Omega$ .

**Theorem 25** Assume  $\Omega$  is a **bounded** domain in  $\mathbb{R}^{n+1}$  with  $t \geq 0$  in  $\Omega$ , and there is a positive constant  $k$  such that

$$c \leq k \quad \text{in } \Omega \quad \text{and} \quad \beta^0 < 0 \quad \text{on } \mathcal{P}\Omega. \quad (39)$$

If  $u \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$  with  $Lu \geq 0$  in  $\Omega$  and  $Mu \geq \beta^0\Phi$  on  $\mathcal{P}\Omega$  for some nonnegative constant  $\Phi$ , then

$$u(X) \leq e^{kt}\Phi \quad \text{for all } X \in \Omega. \quad (40)$$

**Proof.** Let  $v = u - e^{kt}\Phi$ . Then  $v \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$  with

$$Lv = a^{ij}(X)D_{ij}v + b^i(X)D_iv + c(X)v - v_t = Lu - (c - k)e^{kt}\Phi \geq 0 \quad \text{in } \Omega$$

and (note that  $t \geq 0$  in  $\Omega$  and so  $e^{kt} \geq 1$  in  $\Omega$ )

$$\begin{aligned} Mv &= Mu - M(e^{kt}\Phi) = Mu - [\beta^{n+1}\partial_t(e^{kt}\Phi) + \beta^0(e^{kt}\Phi)] \quad \text{on } \mathcal{P}\Omega \\ &\geq \beta^0(1 - e^{kt})\Phi \geq 0 \quad \text{on } \mathcal{P}\Omega. \end{aligned}$$

The proof is done by Lemma 23. □

**Corollary 26** (*comparison principle; NO sign condition on  $c$* ) Assume (39). If  $u, v \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$  with  $Lu \geq Lv$  in  $\Omega$  and  $Mu \geq Mv$  on  $\mathcal{P}\Omega$ , then  $u \leq v$  in  $\Omega$ . In particular, if  $Lu = Lv$  in  $\Omega$  and  $Mu = Mv$  on  $\mathcal{P}\Omega$ , then  $u \equiv v$  in  $\Omega$ .

**Remark 27** (*important*) By taking  $\beta \equiv 0$  and  $\beta^0 \equiv -1$ , we have that  $Lu \geq Lv$  in  $\Omega$  and  $u \leq v$  on  $\mathcal{P}\Omega$  implies  $u \leq v$  in  $\Omega$ . **Note that we have NO sign condition on  $c$  here.**

We shall give some applications of the above weak maximum principle.

**Theorem 28** Assume  $c \leq k$  in  $\Omega$  and  $I(\Omega) \subset (0, T)$ , where

$$I(\Omega) = \{t \in \mathbb{R} : (x, t) \in \Omega \text{ for some } x \in \mathbb{R}^n\}.$$

If  $u \in C^{2,1}(\Omega) \cap C^0(\bar{\Omega})$  with  $Lu \geq f$  in  $\Omega$ , then

$$\sup_{\Omega} u \leq e^{(k+1)T} \left( \sup_{\mathcal{P}\Omega} u^+ + \sup_{\Omega} f^- \right). \quad (41)$$

If  $Lu = f$  in  $\Omega$ , then

$$\sup_{\Omega} |u| \leq e^{(k+1)T} \left( \sup_{\mathcal{P}\Omega} |u| + \sup_{\Omega} |f| \right) \quad (42)$$

where  $a^+ = \max\{a, 0\} \geq 0$ ,  $a^- = \max\{-a, 0\} \geq 0$ .

**Proof.** If  $Lu \geq f$ , set  $v = e^{(k+1)t}$  (we have  $v \geq 1$  in  $\Omega$ ) and

$$F = \sup_{\mathcal{P}\Omega} u^+ + \sup_{\Omega} f^- \quad (F \geq 0 \text{ is a constant}).$$

We have

$$Lv = [c - (k+1)]v \leq -v \leq -1 \quad \text{in } \Omega.$$

Hence

$$L(u - Fv) \geq f - F(Lv) \geq f + F = f + \sup_{\mathcal{P}\Omega} u^+ + \sup_{\Omega} f^- \geq 0 \quad \text{in } \Omega$$

and

$$u - Fv \leq u - F = u - \left( \sup_{\mathcal{P}\Omega} u^+ + \sup_{\Omega} f^- \right) \leq 0 \quad \text{on } \mathcal{P}\Omega.$$

The proof is done by Remark 27.

If  $Lu = f$  (this implies  $Lu \geq f$  and  $L(-u) \geq -f$ ), then we have both

$$\sup_{\Omega} u \leq e^{(k+1)T} \left( \sup_{\mathcal{P}\Omega} u^+ + \sup_{\Omega} f^- \right)$$

and

$$\sup_{\Omega} (-u) \leq e^{(k+1)T} \left( \sup_{\mathcal{P}\Omega} (-u)^+ + \sup_{\Omega} (-f)^- \right) = e^{(k+1)T} \left( \sup_{\mathcal{P}\Omega} u^- + \sup_{\Omega} f^+ \right)$$

which gives (42). □

**Remark 29** If  $Lu \geq 0$  in  $\Omega$  and  $c \leq 0$  in  $\Omega$ , then we just choose

$$F = \sup_{\mathcal{P}\Omega} u^+ \quad (F \geq 0 \text{ is a constant})$$

and compute

$$L(u - F) \geq 0 - cF \geq 0 \quad \text{in } \Omega$$

and

$$u - F = u - \left( \sup_{\mathcal{P}\Omega} u^+ \right) \leq 0 \quad \text{on } \mathcal{P}\Omega.$$

Therefore  $u \leq F$  in  $\Omega$ , which gives the better estimate

$$\sup_{\Omega} u \leq \sup_{\mathcal{P}\Omega} u^+. \quad (43)$$

More generally, we have the following better estimate in case  $c \leq 0$  in  $\Omega$ .

**Theorem 30** (*weak maximum principle for  $c \leq 0$* ) Assume  $c \leq 0$  in  $\Omega$  and  $|x| \leq R$  for some positive constant  $R$  for all  $X = (x, t) \in \Omega$ . Also assume there exist positive constants  $\lambda, \Lambda$  such that

$$\begin{cases} a^{ij}(X) \xi_i \xi_j \geq \lambda |\xi|^2, & \forall \xi \in \mathbb{R}^n, \quad X \in \Omega \\ |b^i(X)| \leq \Lambda, & \forall i = 1, \dots, n, \quad X \in \Omega. \end{cases} \quad (44)$$

If  $Lu \geq f$  in  $\Omega$ , then

$$\sup_{\Omega} u \leq \sup_{\mathcal{P}\Omega} u^+ + C(\lambda, \Lambda, R) \sup_{\Omega} f^- \quad (45)$$

and if  $Lu = f$  in  $\Omega$ , then

$$\sup_{\Omega} |u| \leq \sup_{\mathcal{P}\Omega} |u| + C(\lambda, \Lambda, R) \sup_{\Omega} |f| \quad (46)$$

where  $C(\lambda, \Lambda, R)$  is a positive constant depending on  $\lambda, \Lambda$ , and  $R$ .

**Remark 31** Note the inequality for arbitrary function  $f$  on  $\Omega$  :

$$f + \sup_{\Omega} f^- \geq 0 \quad \text{on } \Omega$$

and

$$f - \sup_{\Omega} f^+ \leq 0 \quad \text{on } \Omega.$$

**Proof.** Let  $d = d(R) > 0$  be the width of  $\Omega$  in the  $x^1$ -direction. That is:  $d$  is the smallest positive number such that, for some  $x_0^1$

$$x_0^1 - d \leq x^1 \leq x_0^1 \quad (47)$$

for all  $x^1$  with  $(x^1, x^2, \dots, x^n, t) \in \Omega$ . Choose  $\alpha = \alpha(\lambda, \Lambda) > 0$  so that

$$a^{11}(X)\alpha^2 + b^1(X)\alpha \geq 1 \quad \text{for all } X \in \Omega$$

and set

$$v(X) = 1 - e^{\alpha(x^1 - x_0^1)} \geq 0 \quad \text{in } \Omega$$

where  $v(X)$  is smooth in  $\bar{\Omega}$ . We have

$$1 - e^{-\alpha d} \geq 1 - e^{\alpha(x^1 - x_0^1)} \geq 0 \quad \text{in } \Omega$$

and (note that  $c \leq 0$ ,  $v \geq 0$ )

$$(Lv)(X) = -[a^{11}(X)\alpha^2 + b^1(X)\alpha]e^{\alpha(x^1 - x_0^1)} + cv \leq -e^{\alpha(x^1 - x_0^1)} \leq -e^{-\alpha d} \quad \text{in } \Omega. \quad (48)$$

Consider the function

$$w(X) = u(X) - \sup_{\mathcal{P}\Omega} u^+ - \left( e^{\alpha d} \sup_{\Omega} f^- \right) v(X), \quad X \in \Omega.$$

We see that

$$\begin{aligned} (Lw)(X) &= (Lu)(X) - c(X) \sup_{\mathcal{P}\Omega} u^+ - \left( e^{\alpha d} \sup_{\Omega} f^- \right) (Lv)(X) \\ &\geq f(X) + \left( e^{\alpha d} \sup_{\Omega} f^- \right) e^{-\alpha d} = f(X) + \sup_{\Omega} f^- \geq 0 \quad \text{in } \Omega \end{aligned}$$

and

$$w(X) = u(X) - \sup_{\mathcal{P}\Omega} u^+ - \left( e^{\alpha d} \sup_{\Omega} f^- \right) v(X) \leq 0 \quad \text{on } \mathcal{P}\Omega.$$

The proof is done by the weak maximum principle.

The proof of (46) is similar. □