Abstract

During the course Analysis II in NTU 2018 Spring, this solution file is latexed by the teaching assistant Yung-Hsiang Huang\(^1\) with the discussions or help from the following contributors:

Exercise 3-5 He-qing Huang; Exercise 7- Mighty Yeh; Exercise 10- ???; Exercise 11- ???; Exercise 12- ???; Exercise 14- ???; Exercise 15- ???; Exercise 16- ???; Problem 1- ???; Problem 2- ???; Problem 3- ???; Problem 4- ???;

1 Exercises

1. Prove that there are infinitely many primes by observing that there were only finitely many \(p_1, \ldots, p_N\), then

\[
\prod_{j=1}^{N} \frac{1}{1 - 1/p_j} \geq \sum_{n=1}^{\infty} \frac{1}{n}
\]

Proof. This is a simple consequence of Theorem 1.6. \(\square\)

2. In the text we showed that there are infinitely many primes of the form \(4k+3\) by a modification of Euclid’s original argument. One can easily adapt this technique to prove the similar result for primes of the form \(3k+2\), and for those of the form \(6k+5\).

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3. Using the same map as Problem 1 of Chapter 7 one can prove that if \( m \) and \( n \) are relatively prime, then \( \mathbb{Z}^*(m) \times \mathbb{Z}^*(n) \) is isomorphic to \( \mathbb{Z}^*(mn) \). For surjectivity (say, given \( (a,b) \in \mathbb{Z}^*(m) \times \mathbb{Z}^*(n) \)), one has to verify \( k = bmx + any \in \mathbb{Z}^*(mn) \) where \( mx + ny = 1 \) (comes from Corollary 1.3). This can be verified as follows: suppose not, say there is a prime \( p \mid k \) and \( p \mid m \), then \( p \mid a \) since \( p \not| ny \) and hence contradicts to the fact \( a \in \mathbb{Z}^*(m) \).

4. Let \( \varphi(n) \) denote the number of positive integers \( \leq n \) that are relatively prime to \( n \). Use the order of groups in the previous exercise, one knows that if \( n \) and \( m \) are relatively prime, then 
\[
\varphi(mn) = \varphi(n) \varphi(m).
\]
Moreover, one can give a formula for Euler phi-function as follows:

(a) Calculate \( \varphi(p) \) when \( p \) is a prime by counting the number of elements in \( \mathbb{Z}^*(p) \).

(b) Give a formula for \( \varphi(p^k) \) when \( p \) is a prime and \( k \geq 1 \) by counting the number of elements in \( \mathbb{Z}^*(p^k) \).

(c) Show that 
\[
\varphi(n) = n \prod_i \left(1 - \frac{1}{p_i}\right)
\]
where \( p_i \) are the primes that divide \( n \).

Proof. (a) \( \varphi(p) = p - 1 \) if \( p \) is a prime.

(b) Claim: \( \varphi(p^k) = p^k - p^{k-1} \) for \( k \geq 1 \). This can be proved as follows: if \( p \mid s \), then \( s \not\in \mathbb{Z}^*(p^k) \). On the other hand, if \( p \not| s \), since \( p \) is a prime, \( s \in \mathbb{Z}^*(p^k) \). So \( \varphi(p^k) = p^k - p^{k-1} \), the order of \( \mathbb{Z}(p^k) \) minus the number of multiples of \( p \) that less than \( p^k \).

(c) By the multiplicative property of \( \varphi \) and (b), 
\[
\varphi(n) = \varphi(p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}) = \varphi(p_1^{a_1})\varphi(p_2^{a_2}) \cdots \varphi(p_k^{a_k}) = p_1^{a_1}(1 - \frac{1}{p_1}) \cdots p_k^{a_k}(1 - \frac{1}{p_k}) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right).
\]

5. If \( n \) is a positive integer, show that 
\[
n = \sum_{d|n} \varphi(d),
\]
where \( \varphi \) is the Euler phi-function.

[Hint: There are precisely \( \varphi(n/d) \) integers \( 1 \leq m \leq n \) with \( \gcd(m,n) = d \).]

Proof. Note that 
\[
\left\{ \frac{i}{n} : 1 \leq i \leq n \right\} = \bigcup_{d|n} \left\{ \frac{j}{d} : 1 \leq j \leq d, \gcd(d,j) = 1 \right\} =: \bigcup_{d|n} A_d
\]
and \( \{A_d\}_{d \in \mathbb{N}} \) are pairwisely disjoint. So one completes the proof by computing the cardinality of sets in both sides.

6. **Write down the characters of the groups** \( \mathbb{Z}^\ast(3), \mathbb{Z}^\ast(4), \mathbb{Z}^\ast(5), \mathbb{Z}^\ast(6), \) **and** \( \mathbb{Z}^\ast(8) \).

(a) Which ones are real, or complex?

(b) Which ones are even, or odd? (A character is even if \( \chi(-1) = 1 \), and odd otherwise).

**Proof.** Since \( \mathbb{Z}^\ast(3), \mathbb{Z}^\ast(4), \) and \( \mathbb{Z}^\ast(6) \) are all \( \cong \mathbb{Z}(2) = \{0, 1\} \), their characters contain the trivial one and the one \( \chi(0) = 1, \chi(1) = -1 \) only, both are real and even.

For \( \mathbb{Z}^\ast(5) \cong \mathbb{Z}(4) = \{0, 1, 2, 3\} \). The characters are \( \chi_j(k) = e^{2\pi i jk} (j, k = 0, 1, 2, 3) \). So \( \chi_0, \chi_2 \) are real. \( \chi_1, \chi_3 \) are complex. Only \( \chi_0 \) is even.

For \( \mathbb{Z}^\ast(8) \cong \mathbb{Z}(2) \times \mathbb{Z}(2) = \{(0,0), (1,0), (0,1), (1,1)\} \). Because of \( (1,0) + (1,0) = (0,0) \), \( \chi((1,0)) = \pm 1 \) for each character \( \chi \). Same for \( (0,1) \) and \( (1,1) \). So every character is real. Note that \( A + B = C \) for \( \{A, B, C\} = \{(1,0), (0,1), (1,1)\} \), so \(-1 \) appears twice or never appears in the values that each character takes at \( \{A, B, C\} \). Hence the even character are the trivial one and the one \( \chi((1,1)) = \chi((0,0)) = 1 \) and \( \chi((1,0)) = \chi((0,1)) = -1 \).

7. **Recall that for** \( |z| < 1 \),

\[
\log_1 \left( \frac{1}{1-z} \right) = \sum_{k \geq 1} \frac{z^k}{k}.
\]

We have seen that

\[
e^{\log_1 \left( \frac{1}{1-z} \right)} = \frac{1}{1-z}.
\]

(a) **Show that if** \( w = 1/(1-z) \), **then** \( |z| < 1 \) **if and only if Re\( (w) > 1/2 \).**

(b) **Show that if** Re\( (w) > 1/2 \) **and** \( w = \rho e^{i\varphi} \) **with** \( \rho > 0, |\varphi| < \pi \), **then**

\[
\log_1 w = \log \rho + i\varphi.
\]

[Hint: If \( e^\zeta = w \), then the real part of \( \zeta \) is uniquely determined and its imaginary part is determined modulo \( 2\pi \).]

**Remark** 1. (a) is the Möbius transformation.

**Proof.** (a) can be proved by brutal computations and Arithmetic-Geometric Means inequality.

(b) As hint, \( e^{\log \rho + i\varphi} = \rho e^{i\varphi} = w = \frac{1}{1-z} \) for some \( |z| < 1 \) from (a). Then

\[
e^{\log \rho + i\varphi} = \frac{1}{1-z} = e^{\log_1 \left( \frac{1}{1-z} \right)} = e^{\log_1 w}.
\]
8. Let $\zeta$ denote the zeta function defined for $s > 1$.

(a) Compare $\zeta(s)$ with $\int_1^\infty x^{-s} \, dx$ to show that

$$\zeta(s) = \frac{1}{s-1} + O(1) \quad \text{as } s \to 1^+.$$ 

(b) Prove as a consequence that

$$\sum_p \frac{1}{p^s} = \log\left(\frac{1}{s-1}\right) + O(1) \quad \text{as } s \to 1^+.$$

Proof. (a) Use mean-value theorem, one has

$$|\zeta(s) - \int_1^\infty \frac{1}{x^s} \, dx| = \left| \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} \, dx \right| = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s} \, dx \leq \sum_{n=1}^{\infty} \frac{s}{n^{s+1}}.$$

(b) is a consequence of (a) and the fact $\log \zeta(s) = \sum_p \frac{1}{p^s} + O(1)$ proved in Proposition 1.11.

9. Let $\chi_0$ denote the trivial Dirichlet character mod $q$, and $p_1, \ldots, p_k$ the distinct prime divisors of $q$. Recall that $L(s, \chi_0) = (1-p_1^{-s}) \cdots (1-p_k^{-s}) \zeta(s)$, and show as a consequence

$$L(s, \chi_0) = \frac{\varphi(q)}{q} \frac{1}{s-1} + O(1) \quad \text{as } s \to 1^+$$

Proof. Note that, by Exercise 8 and mean-value theorem to $f(s) = \prod_{j=1}^k (1-p_j^{-s})$,

$$L(s, \chi_0) = \prod_{j=1}^k (1-p_j^{-s}) \zeta(s) = \left[ \prod_{j=1}^k (1-p_j^{-s}) - \prod_{j=1}^k (1-p_j) \right] \zeta(s) + \frac{\varphi(q)}{q} \zeta(s)$$

$$= O(s-1)(\frac{1}{s-1} + O(1)) + \frac{\varphi(q)}{q} \frac{1}{s-1} + O(1).$$

10. Show that if $l$ is relatively prime to $q$, then

$$\sum_{p \equiv l} \frac{1}{p^s} = \frac{1}{\varphi(q)} \log\left(\frac{1}{s-1}\right) + O(1) \quad \text{as } s \to 1^+.$$

This is a quantitative version of Dirichlet’s Theorem.

Proof.
11. Use the characters for $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5)$, and $\mathbb{Z}^*(6)$ to verify directly that $L(1, \chi) \neq 0$ for all non-trivial Dirichlet characters modulo $q$ when $q = 3, 4, 5,$ and $6$.

[Hint: Consider in each case the appropriate alternating series.]

Proof.

12. Suppose $\chi$ is real and non-trivial; assuming the theorem that $L(1, \chi) \neq 0$, show directly that $L(1, \chi) > 0$.

[Hint: Use the product formula for $L(s, \chi)$.]

Proof.

13. Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_n = a_m$ if $n = m \mod q$. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges if and only if $\sum_{n=1}^{q} a_n = 0$.

[Hint: Summation by parts.]

Proof. Let $A_j = \sum_{k=1}^{j} a_k$ with convention that $A_0 = 0$. Recall that

$$\sum_{n=1}^{N} \frac{a_n}{n} = \sum_{n=1}^{N} A_n \frac{1}{n(n+1)} + \frac{A_N}{N+1}$$

The periodicity implies that ($[x]$ is the floor function of $x$)

$$A_N = A_q [\frac{N}{q}] + O(1).$$

So the second term is always bounded. Moreover, the first term converges if and only if $A_q = 0$.

14. The series

$$F(\theta) = \sum_{|n| \neq 0} \frac{e^{i n \theta}}{n}, \text{ for } |\theta| < \pi,$$

converges for every $\theta$ and is the Fourier series of the function defined on $[-\pi, \pi]$ by $F(0) = 0$ and

$$F(\theta) = \begin{cases} i(-\pi - \theta) & \text{if } -\pi \leq \theta < 0 \\ i(\pi - \theta) & \text{if } 0 < \theta \leq \pi, \end{cases}$$

and extended by periodicity (period $2\pi$) to all of $\mathbb{R}$ (see Exercise 8 in Chapter 2).
Show also that if $\theta \neq 0 \mod 2\pi$, then the series
\[ E(\theta) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} \]
converges, and that
\[ E(\theta) = \frac{1}{2} \log \left( \frac{1}{2 - 2\cos \theta} \right) + \frac{i}{2} F(\theta) \]

\[ \text{Proof.} \]

15. To sum the series $\sum_{n=1}^{\infty} a_n/n$ with $a_n = a_m$ if $n = m \mod q$ and $\sum_{n=1}^{q} a_n = 0$, proceed as follows. (a) Define
\[ A(m) = \sum_{n=1}^{q} a_n \zeta^{-mn} \text{ where } \zeta = e^{2\pi i/q} \]
Note that $A(q) = 0$. With the notation of the previous exercise, prove that
\[ \sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{q} \sum_{m=1}^{q-1} A(m) E(2\pi m/q). \]
[Hint: Use Fourier inversion on $\mathbb{Z}(q)$.]

(b) If $\{a_m\}$ is odd, $(a_{-m} = -a_m)$ for $m \in \mathbb{Z}$, observe that $a_0 = a_q = 0$ and show that
\[ A(m) = \sum_{1 \leq n < q/2} a_n (\zeta^{-mn} - \zeta^{mn}). \]

(c) Still assuming that $\{a_m\}$ is odd, show that
\[ \sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{2q} \sum_{m=1}^{q-1} A(m) F(2\pi m/q). \]
[Hint: Define $\tilde{A}(m) = \sum_{n=1}^{q} a_n \zeta^{mn}$ and apply the Fourier inversion formula.]

\[ \text{Proof.} \]

16. Use the previous exercises to show that
\[ \frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots, \]
which is $L(1, \chi)$ for the non-trivial (odd) Dirichlet character modulo 3.

\[ \text{Proof.} \]
2 Problems

1. Here are other series that can be summed by the methods in (a) For the non-trivial Dirichlet character modulo 6, $L(1, \chi)$ equals

\[
\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots,
\]

(b) If $\chi$ is the odd Dirichlet character modulo 8, then $L(1, \chi)$ equals

\[
\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots,
\]

(c) For an odd Dirichlet character modulo 7, $L(1, \chi)$ equals

\[
\frac{\pi}{\sqrt{7}} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \cdots,
\]

(d) For an even Dirichlet character modulo 8, $L(1, \chi)$ equals

\[
\frac{\log(1 + \sqrt{2})}{\sqrt{2}} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots,
\]

(e) For an even Dirichlet character modulo 5, $L(1, \chi)$ equals

\[
\frac{2}{\sqrt{5}} \log \left( \frac{1 + \sqrt{5}}{2} \right) = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{11} + \cdots,
\]

Proof. 

2. Let $d(k)$ denote the number of positive divisors of $k$. (a) Show that if $k = p_1^{a_1} \cdots p_n^{a_n}$ is the prime factorization of $k$, then

\[d(k) = (a_1 + 1) \cdots (a_n + 1).\]

Although Theorem 3.12 shows that on "average" $d(k)$ is of the order of $\log k$, prove that the following on the basis of (a):

(b) $d(k) = 2$ for infinitely many $k$.

(c) For any positive integer $N$, there is a constant $c > 0$ so that $d(k) \geq c(\log k)^N$ for infinitely many $k$. [Hint: Let $p_1, \cdots, p_N$ be $N$ distinct primes, and consider $k$ of the form $(p_1 p_2 \cdots p_N)^m$ for $m = 1, 2, \cdots$.]

Proof. (a)(b) are easy. (c)
3. Show that if \( p \) is relatively prime to \( q \), then
\[
\prod_{\chi} \left(1 - \frac{\chi(p)}{p^s}\right) = \left(\frac{1}{1 - p^s}\right)^g,
\]
where \( g = \varphi(q)/f \), and \( f \) is the order of \( p \) in \( \mathbb{Z}^*(q) \) (that is, the smallest \( n \) for which \( p^n \equiv 1 \mod q \)). Here the product is taken over all Dirichlet characters modulo \( q \).

**Proof.**

4. Prove as a consequence of the previous problem that
\[
\prod_{\chi} L(s, \chi) = \sum_{n \geq 1} \frac{a_n}{n^s},
\]
where \( a_n \geq 0 \), and the product is over all Dirichlet characters modulo \( q \).

**Proof.**