1. Let \( \nu \) be a complex measure on \( (X, \mathcal{M}) \). If \( E \in \mathcal{M} \), define

\[
\mu_1(E) = \sup \left\{ \sum_{1}^{n} |\nu(E_j)| : n \in \mathbb{N}, E_1, E_2, \ldots E_n \text{ disjoint}, E = \bigcup_{1}^{n} E_j \right\}
\]

\[
|\nu|(E) = \sup \left\{ \sum_{1}^{\infty} |\nu(E_j)| : E_1, E_2, \ldots \text{ disjoint}, E = \bigcup_{1}^{\infty} E_j \right\}
\]

\[
\mu_3(E) = \sup \left\{ |\int_{E} f d\nu| : |f| \leq 1 \right\}
\]

Show that \( \mu_1 = |\nu| = \mu_3 \).

**Remark** 1. We extend the problem slightly, which comes from [3, Exercise 3.21]. To prove the original problem, we can simplify the proof given here, that is, without \( \mu_3 \).

**Proof.** We are going to show \( \mu_1 \leq |\nu| \leq \mu_3 \leq \mu_1 \). The first inequality is trivial. For the second one, since there exists \( w_j \in \mathbb{C}, |w_j| = 1 \) such that \( |\nu(E_j)| = w_j \nu(E_j) \). Consider the function \( f = \sum_j w_j \chi_{E_j} \). Since \( \{E_j\} \) is mutually disjoint, \( |f| \leq 1 \). Take \( f_n = \sum_{1}^{n} w_j \chi_{E_j} \). We then have, by dominate convergence theorem,

\[
|\int_{E} f_n - \int_{E} f d\nu| \leq \int_{E} |f_n - f| d|\nu| \to 0,
\]

We then have

\[
\int_{E} f d\nu = \lim_{n \to \infty} \int_{E} f_n d\nu = \lim_{n \to \infty} \sum_{1}^{n} w_j \nu(E_j) = \sum_{1}^{\infty} |\nu(E_j)|.
\]

Then \( \sum_{1}^{\infty} |\nu(E_j)| = |\int_{E} f d\nu| \leq \mu_3(E) \). Since \( \{E_j\} \) is arbitrary, \( |\nu| \leq \mu_3 \). For the third inequality, given \( \epsilon > 0 \), then we can find some \( f \) with \( |f| \leq 1 \) such that

\[
\mu_3(E) < |\int_{E} f d\nu| + \epsilon
\]
We approximate \( f \) by a simple function as follows. Let \( D \subset \mathbb{C} \) be the closed unit disc. The compactness of \( D \) implies that there are finite many \( z_j \in D \) such that \( B_\varepsilon(z_j) \) covers \( D \). Define \( B_j = f^{-1}(B_\varepsilon(z_j)) \subset X \), which is measurable since \( f \) is. The union of \( B_j \) is \( X \). Let

\[
A_1 = B_1, \ A_j = B_j \setminus \bigcup_{i=1}^{j-1} B_i
\]

be the disjoint sets with \( A_j \subset B_j \) and \( \cup A_j = X \). Define the simple function \( \phi = \sum_1^m z_j \chi_{A_i} \), then \( |\phi| \leq 1 \) and \( |f(x) - \phi(x)| < \epsilon \) for all \( x \) by the construction. Then

\[
|\int_E f \, d\nu| - |\int_E \phi \, d\nu| \leq |\int_E f - \phi \, d\nu| < |\nu|(E).
\]

Now we define \( F_j = A_j \cap E \), then \( F_j \) is a finite partition of \( E \) and

\[
|\int_E \phi \, d\nu| = |\int \sum_j z_j \chi_{A_j \cap E} \, d\nu| = |\sum_j z_j \nu(A_j \cap E)| \leq \sum_j |\nu(F_j)| \leq \mu_1(E).
\]

Since \( \epsilon \) is arbitrary, we have \( \mu_3(E) \leq \mu_1(E) \). \( \square \)

2. Find two measures \( \mu, \lambda \) that are \( \sigma \)-finite positive measure on some \( (X, \mathcal{M}) \), but the function \( h \) in the absolutely continuous part of the Lebesgue-Radon-Nikodym decomposition of \( \lambda \) with respect to \( \mu \) is not \( \mu \)-integrable.

**Proof.** Let \( \mu \) be the Lebesgue measure on \( ((0,1), \mathcal{L}) \). The desired \( \lambda \) is defined by

\[
\lambda(E) = \int_E \frac{1}{x} \, d\mu(x).
\]

which is easy to see it’s positive, \( \sigma \)-finite, absolutely continuous with respect to \( \mu \). \( \square \)

**Remark** 2. This example also provide us a reason why we need to assume the measure \( \lambda \) is finite in Theorem 6.11.

3. **Proof.** Any multiple of complex regular Borel measure is still a complex regular Borel measure.

Given two complex regular Borel measures \( \mu_1, \mu_2 \). Given a Borel set \( E \) and \( k \in \mathbb{N} \), then there exist open sets \( O_1, O_2 \) containing \( E \) such that \( |\mu_i|(O_i \setminus E) < \frac{1}{2k} \) \( (i = 1, 2) \), then

\[
|\mu_1 + \mu_2|(O_1 \setminus E) = |\mu_1 + \mu_2|(E) + |\mu_1 + \mu_2|((O_1 \setminus O_2) \setminus E) \leq |\mu_1 + \mu_2|(E) + |\mu_1|(O_1 \setminus E) + |\mu_2|(O_2 \setminus E) \leq |\mu_1 + \mu_2|(E) + \frac{1}{k},
\]

which implies the outer regularity of \( |\mu_1 + \mu_2| \).

Given any open set \( E \) or Borel set \( E \) with finite measure, we see for each \( k \in \mathbb{N} \), then there exist open sets \( K_1, K_2 \) contained in \( E \) such that \( |\mu_i|(E \setminus K_i) < \frac{1}{2k} \) \( (i = 1, 2) \), then \( |\mu_1 + \mu_2|(K_1 \cup K_2) = \)
\[|\mu_1 + \mu_2|(E) - |\mu_1 + \mu_2|(E \setminus (K_1 \cup K_2)) \geq |\mu_1 + \mu_2|(E) - \frac{1}{k},\] which implies the inner regularity of \(|\mu_1 + \mu_2|\). Hence \(\mu_1 + \mu_2\) is a complex regular Borel measure.

Therefore \(M(X)\) is a vector space over \(\mathbb{C}\). For each \(\nu \in M(X)\), by Radon-Nikodym theorem, there is an integrable function \(f\) with respect to \(\mu = |\nu_r| + |\nu_l|\) such that \(d\nu = f d\mu\) and then \(d|\nu| = |f| d\mu\). So \(|\nu| (X) = \|\nu\| = 0 \iff f\) is zero \(\mu\)-a.e. \(\iff \nu = 0\). For each \(a \in \mathbb{C}\), \(\|a\nu\| = |a| |\nu| (X) = \int_X |a f| d\mu = |a| \|\nu\| (X) = |a| \|\nu\|\). The triangle inequality is proved by the same method with \(\mu = \mu_1 + \mu_2 := |\nu_{1,r}| + |\nu_{1,l}| + |\nu_{2,r}| + |\nu_{2,l}|\). Therefore, \(\| \cdot \|\) is a norm on \(M(X)\).

Given \(\sum \nu_n\) be a absolutely convergent series in \(M(X)\). Since for each \(A \in \Sigma\), \(\sum |\nu_n(A)| \leq \sum |\nu_n|(A) \leq \sum |\nu_n| (X) = \sum \|\nu_n\| < \infty\), we may define \(\nu : \Sigma \to \mathbb{C}\) by \(\nu(A) = \sum \nu_n(A)\). Clearly, \(\nu(\emptyset) = 0\). If \(A_k\) is a sequence of disjoint measurable sets, since by Tonelli’s theorem

\[\sum_{n,k} |\nu_n|(A_k) = \sum_n \sum_k |\nu_n|(A_k) = \sum_n |\nu_n|(\cup_k A_k) \leq \sum_n \|\nu_n\| < \infty\]

and therefore

\[\nu(\cup_k A_k) = \sum_n \nu_n(\cup_k A_k) = \sum_n \sum_k \nu_n(A_k) = \sum_k \sum_n \nu_n(A_k) = \sum_k \nu(A_k).
\]

Hence, \(\nu\) is a complex measure. \(\nu \in M(X)\). Given \(X_1 \cdots X_m\) covers \(X\),

\[\sum_{j=1}^m \left| \left( \nu - \sum_{n=1}^N \nu_n \right) (X_j) \right| = \sum_{j=1}^m \left| \sum_{n=N+1}^\infty \nu_n (X_j) \right| \leq \sum_{j=1}^m \sum_{n=N+1}^\infty |\nu_n|(X_j) = \sum_{n=N+1}^\infty |\nu_n|(X) = \sum_{n=N+1}^\infty \|\nu_n\|.
\]

By Exercise 1, \(\|\nu - \sum_{n=1}^N \nu_n\| \leq \sum_{n=N+1}^\infty \|\nu_n\| \to 0\) as \(N \to 0\). So \(M(X)\) is complete.

4. A special case was discussed in Exercise 4.7 earlier.

1st proof. Suppose there is no \(C > 0\), such that \(\|fg\|_1 \leq C \|f\|_p\). Then there exist \(f_n\) such that \(\|f_n\|_p = 1\) and \(\int |f_n g| > 3^n\). Let \(f = \sum 2^{-n} |f_n|\), then \(\|f\|_p \leq 1\) and for each \(n\)

\[\int |fg| > \int |f_n g| 2^{-n} > \left(\frac{3}{2}\right)^n\]

This contradicts to \(fg \in L^1\) and therefore the map \(f \mapsto \int f g\) from \(L^p \to \mathbb{R}\) or \(\mathbb{C}\) is bounded. If \(1 \leq p < \infty\), then by Riesz Representation theorem, \(g = \tilde{g} \in L^q\) a.e.. For \(p = \infty\), if \(g\) is not integrable, then it contradicts to the hypothesis by taking \(f \equiv 1\).

2nd proof. This is inspired by Exercise 5.10, which is an application of Banach-Steinhaus Theorem. I think this method is easier than the first proof since we don’t need duality theorem.
This time, we define a linear map $\Lambda_n : L^p \to \mathbb{C}$ by $\Lambda_n(f) = \int f g_n$, where $g_n = g \chi_{\{|x| \leq n; |g(x)| \leq n\}}$ (this is the place we use the $\sigma$-finiteness of $\mu$). Note that $\|\Lambda_n\| = \|g_n\|_q$ is bounded for all $n$ and $\{\Lambda_n(f)\}$ is bounded by $\int \{|fg|\}$ for each $f \in L^p$. Therefore $g \in L^q$ by Banach-Steinhaus Theorem and monotone convergence theorem. 

There is a theorem so called the converse Hölder’s inequality:

**Theorem 3.** Let $(X, \Sigma, \mu)$ be a measure space, $1 \leq q < \infty$. If $g \in L^q$, then

$$\|g\|_q = \sup\{|\int fg\,d\mu| : \|f\|_p = 1\}.$$ 

If $q = \infty$, this result holds if $\mu$ is semi-finite.

This exercise seems to be the converse of the theorem (but it’s NOT) if we assume

$$M_q(g) = \sup\{|\int fg\,d\mu| : f \text{ is simple and its support has finite measure, } \|f\|_p = 1\} < \infty.$$ 

I see this result in Folland [3, Theorem 6.14]. Compare the differences between their proofs. I think they can NOT be replaced by each other. (Reason: Exercise 6.4 do NOT permit $M_q(g) < \infty$. On the other hand, I think we can’t use approximate argument to see $M_q(g) < \infty$ implies $fg \in L^1$ for all $f \in L^p$.)

5. **Proof.** No, since $L^1(\mu)$ is one dimensional, so does its dual space. But $L^\infty(\mu)$ is two dimensional.

**Remark** 4. This exercise shows that the map from $L^\infty \to (L^1)^*$ defined by $g \mapsto \phi_g$ is not injective if $\mu$ is not semifinite. This problem, however, can be remedied by redefining $L^\infty$. See Folland [3, Exercise 6.23-25].

6. **Proof.** For each $\sigma$-finite set $E \subset X$ there is an a.e.-unique $g_E \in L^q(E)$ such that $\Phi(f) = \int f g_E$ for all $f \in L^p(E)$ and $\|g_E\|_q \leq \|\Phi\|$. If $F$ is $\sigma$-finite and $E \subset F$, then $g_F = g_E$ a.e. on $E$, so $\|g_F\|_q \geq \|g_E\|_q$. Let $M$ be the supremum of $\|g_E\|_q$ as $E$ ranges over all $\sigma$-finite sets, noting that $M \leq \|\Phi\|$. Choose a sequence $\{E_n\}$ so that $\|g_{E_n}\|_q \to M$, and set $F = \bigcup_1^\infty E_n$. Then $F$ is $\sigma$-finite and $\|g_F\|_q \geq \|g_{E_n}\|_q$ for all $n$, whence $\|g_F\|_q = M$. Now, if $A$ is a $\sigma$-finite set containing $F$, we have

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M_q = \int |g_F|^q,$$

and thus $g_{A\setminus F} = 0$ and $g_A = g_F$ a.e. (Here we use the fact $q < \infty$.) But if $f \in L^p$, then $A_f = F \cup \{x : f(x) \neq 0\}$ is $\sigma$-finite, so $\Phi(f) = \int f g_{A_f} = \int f g_F$. Thus we may take $g = g_F$, and the proof is complete.
7. Proof. As Hint, the hypothesis holds for any \( d\mu \) with \( f \) is a trigonometric polynomial, and hence with \( f \) is a continuous function by Weierstrass’ theorem, and therefore with \( f \) is any bounded Borel function by Lusin’s theorem. Since by Theorem 6.12, \( d\mu = h\,d|\mu| \) for some \( |h| = 1 \) a.e., \( |\mu|(n) \to 0 \) as \( n \to \infty \). Since \( |\mu|(n)| = |\mu|(n)| = |\mu|(-n)| \), we know \( |\mu|(m) \to 0 \) as \( m \to -\infty \). By the same argument, it holds with \( f\,d|\mu| \) for any \( f \) is bounded. In particular, we pick \( h\,d|\mu| = d\mu \).

8. Proof. Assume that there is some \( k \in \mathbb{N} \) such that for all \( n \),

\[
\int e^{-int} \, d\nu := \int e^{-i(ekt - 1)} \, d\mu = \hat{\mu}(n + k) - \hat{\mu}(n) = 0
\]

Since trigonometric polynomials are dense in \( C[0, 2\pi] \), \( \int f \, d\nu = 0 \) for any continuous \( f \). Given open set \( U \), there is an increasing sequence of continuous function \( f_n = 1 \), if \( \text{dist}(x, \partial U) > \frac{1}{n} \) and \( f_n = n \cdot \text{dist}(x, \partial U) \) otherwise, that approximate to the characteristic function \( \chi_U \) pointwise. By monotone convergence theorem, we see \( \nu(U) = 0 \). So \( \nu_r(U) = \nu(U) = 0 \), that is \( \nu_r = \nu_i = 0 = \nu \). Since \( (e^{-ikt} - 1) \neq 0 \) iff \( t \neq \frac{2\pi j}{k} (j \in \mathbb{N}) \), \( \mu = \sum_{j=0}^{k-1} a_j \delta_{\frac{2\pi j}{k}} \) for some \( a_j \in \mathbb{C} \).

9. Proof. We are going to construct \( h_n \geq 0 \) such that the conditions (i)(ii) are satisfied and for each \( f \in C(I) \), \( \int fh_n \, dm \to \int f \, dm \), more precisely, the Riemann integral of \( f \). The desired \( g_n = (1 + \frac{1}{n})^{-1}(h_n + \frac{1}{n}) > 0 \) and satisfies conditions (i)-(iii).

Let \( \delta_n = (2n^3)^{-1} \). Consider \( h_n = \sum_{k=1}^{n} h_k^n \) where

\[
h_k^n(x) = \begin{cases} 
2n^2 & \text{for } |x - \frac{2k-1}{2n}| \leq \frac{n-1}{2n}\delta_n, \\
\text{linear} & \text{for } \frac{n-1}{2n}\delta_n < |x - \frac{2k-1}{2n}| \leq \frac{n+1}{2n}\delta_n, \\
0 & \text{for } \frac{1}{2}\delta_n < |x - \frac{2k-1}{2n}|.
\end{cases}
\]

Now observe that it’s easy to check they satisfies (i)(ii). Moreover,

\[
|\int fh_n \, dm - \frac{1}{n} \sum_{k=1}^{n} f\left( \frac{2k-1}{2n} \right)| \\
\leq \sum_{k=1}^{n} \int_{|x - \frac{2k-1}{2n}| \leq \frac{n-1}{2n}\delta_n} 2n^2 |f(x) - f\left( \frac{2k-1}{2n} \right)| \, dx + \left( 2n^2 \|f\|_{\infty} \frac{1}{n}\delta_n + \|f\|_{\infty} \frac{1}{n^2}\right) n \\
\leq \sup_{k=1,\ldots,n} \sup \{ |f\left( \frac{2k-1}{2n} \right) - f(x)| : |x - \frac{2k-1}{2n}| \leq \frac{n-1}{2n}\delta_n \} + \|f\|_{\infty} \frac{2}{n}
\]

which tends to zero as \( n \to 0 \) by the uniform continuity of \( f \). This is what we want to see.

10. (a) it’s obvious.
The Vitali’s Convergence Theorem is easy to prove if we know Egoroff’s theorem, Fatou’s lemma, and the equivalence between the uses of $|\int f|$ and $\int |f|$ in the definition of uniform integrability.

(c) Show that we can not omit the tightness condition (iii): for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for all $n$, even if $\{\|f_n\|_1\}$ is bounded.

**Proof.** Consider the Lebesgue measure on $(-\infty, \infty)$ with $f_n = \chi_{(n,n+1)}$.

(d) To apply Vitali’s theorem in finite measure space, sometimes we see $|f(x)| < \infty$ a.e. is automatically true, but sometimes it’s not. Give examples.

**Proof.** (i) Consider the Lebesgue measure on $[0, 1]$, by the uniform integrability we know there exist nonoverlapping closed intervals $I_1, \cdots I_k$ whose union is $[0, 1]$, and for all $j = 1, \cdots k$ and $n \in \mathbb{N}, \int_{I_j} |f_n| < 1$. (It’s easy to show it’s equivalent to use $|\int f|$ and $\int |f|$ in the definition of uniform integrability.) By Fatou’s Lemma,

$$\int_{[0,1]} |f| \leq \liminf_n \int_{[0,1]} |f_n| < k.$$

(ii) We need to find out a finite measure space $(X, \mathcal{M}, \mu)$ and an uniformly integrable sequence of $L^1$ functions $f_n$ with $f_n \rightarrow f$ a.e., $f(x)$ is not finite a.e. and $f_n \nrightarrow f$ in $L^1$.

On $\mathbb{R}$, let $\mathcal{M}$ is the $\sigma$-algebra of countable or co-countable sets. $\mu(E) := 0$ if $E$ is countable, $\mu(E) := 1$ if $E$ is co-countable, which is easy to check $\mu$ is a measure on $\mathcal{M}$. Consider $f_n \equiv n$, which is the desired example.

(e) It’s easy to see Vitali’s Theorem implies Lebesgue Dominated Convergence Theorem in finite measure space. The sequence $f_n(x) = \frac{1}{x} \chi_{[1/n, 2/n]}(x)$ is an example in which Vitali’s theorem applies although the hypothesis of Lebesgue’s theorem do not hold.

(f) The sequence $f_n = n \chi_{[0,1/n]} - n \chi_{[1-1/n, 1]}$ on $[0,1]$ shows the assumption that $f_n \geq 0$ is sometimes important in some applications. Note that $f_n(x) \rightarrow 0$ for every $x \in [0,1], \int f_n(x) \, dx = 0$, but $f_n$ is not uniformly integrable.

(g) However, the following converse of Vitali’s theorem is true:

**Theorem 5.** If $\mu(X) < \infty$, $f_n \in L^1(\mu)$, and $\lim_{n \rightarrow \infty} \int_E f_n \, d\mu$ exists for every $E \in \mathcal{M}$, then $\{f_n\}$ is uniformly integrable.
Proof. As hint, we define \( \rho(A,B) = \int |\chi_A - \chi_B| \, d\mu \). Then \((\mathcal{M}, \rho)\) is a complete metric space (modulo sets of measure zero), and \( E \mapsto \int_E f_n \, d\mu \) is continuous for each \( n \), (denote this map by \( F_n \)). If \( \epsilon > 0 \), consider \( A_N = \{ E : |F_n(E) - F_m(E)| < \epsilon, \text{ if } n, m \geq N \} \). Since \( X = \cup A_N \) by hypothesis, Baire Category theorem implies that some \( A_N \) has nonempty interior, that is, there exist \( E_0 \in \mathcal{M}, \delta > 0, N \in \mathbb{N} \) so that

\[
|\int_{E_0} (f_n - f_N) \, d\mu| < \epsilon \text{ if } \rho(E, E_0) < \delta, n > N.
\] (2)

If \( \mu(A) < \delta \), (2) holds with \( B = E_0 \setminus A \) and \( C = E_0 \cup A \) in place of \( E \). Thus,

\[
|\int_A (f_n - f_N) \, d\mu| = |\int_C - \int_B (f_n - f_N) \, d\mu| < 2\epsilon.
\]

By considering \( \{f_1, \cdots, f_N\} \), there exists \( \delta' > 0 \), such that

\[
|\int_A f_n \, d\mu| < 3\epsilon \text{ if } \mu(A) < \delta', n = 1, 2, 3, \cdots
\]

Remark 6. The Dunford-Pettis theorem: \([2, \text{p.466 and 472}]\)

11. Proof. By Fatou’s lemma \( \int_X |f|^p < C \) and hence \( f \) is finite a.e.. Given \( \epsilon > 0 \), by Egoroff’s theorem, there is a measurable set \( F \) with \( C^{1/p} \mu(F)^{1-1/p} < \epsilon \), such that \( f_n \to f \) uniformly on \( X \setminus F \). Then there is an \( N(\epsilon) \in \mathbb{N} \) such that for \( n > N(\epsilon) \),

\[
\int_X |f_n - f| \leq \int_{X \setminus F} |f_n - f| + \int_F |f_n| + \int_F |f| < \epsilon + 2C^{1/p} \mu(F)^{1-1/p} < 3\epsilon.
\]

Remark 6. The Dunford-Pettis theorem: \([2, \text{p.466 and 472}]\)

12. The assertions in this problem are easy to prove, we omit the proof and remark several things:

First, this problem shows that the map \( g \mapsto \Phi_g \) from \( L^\infty \to (L^1)^* \) is not surjective if the measure space is not \( \sigma \)-finite; second, that map is also not injective if \( \mu \) is not semifinite, this can be seen from the example I learned from Folland \([3, \text{p.191-192}]\): Let \( E \subset X \) be a set of infinite measure that contains no subset of positive finite measure. Then for any \( f \in L^1 \), the set \( \{x : f(x) \neq 0\} \) intersects \( E \) in a null set. It follows that \( \Phi_{\chi_E} = 0 \) although \( \chi_E \neq 0 \) in \( L^\infty \).

13. Apply Hahn-Banach Theorem, see Theorem 5.19, p.107.

Additional Results

We present the proof that the constant \( \frac{1}{\pi} \) in Sec 6.3 is sharpest. (Due to Bledsoe \([1]\) which is a simpler treatment of Kaufman-Rickert \([4]\)).
Theorem 7. Let $C_0$ be the supremum of the number $C$ for which

$$| \sum_{z \in S} | \geq C \sum_{z \in T} |z|$$

where $T$ is any nonempty finite set of complex numbers and $S$ is any subset of $T$. Then $C_0 = \frac{1}{\pi}$.

Proof. By Theorem 6.3, $C_0 \geq \frac{1}{\pi}$. On the other hand, consider $z_j^n = \exp[i(2\pi/n)j]$, $j = 1, \ldots, n$. For this choice of points we have for each $\theta$, let $S^n_\theta$ be the set of those $z_j^n$ such that $|\theta - \arg(z_j^n)| \leq \frac{\pi}{2}$.

$$\lim_{n \to \infty} \left( | \sum_{z \in S^n_\theta} z | / \sum_{j=1}^n |z_j| \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{z \in S^n_\theta} z$$

$$= \lim_{n \to \infty} \frac{1}{2\pi} \frac{\pi}{n/2} \sum_{z \in S^n_\theta} z$$

$$= \frac{1}{2\pi} \int_{\theta - \pi/2}^{\theta + \pi/2} e^{i\phi} d\phi = \frac{1}{\pi}.$$

Thus, $C_0 \leq \frac{1}{\pi}$.

References


