1. Let $X$ consists of two points $a$ and $b$, put $\mu(\{a\}) = \mu(\{b\}) = \frac{1}{2}$, and let $L^p(\mu)$ be the resulting real $L^p$ space. Identify each real function $f$ on $X$ with the point $(f(a), f(b))$ in the plane, and sketch the unit balls of $L^p(\mu)$, for $0 < p \leq \infty$. Note that they are convex if and only if $1 \leq p \leq \infty$. For which $p$ is this unit ball a square? A circle? If $\mu(\{a\}) \neq \mu(\{b\})$, how does this situation differ from the preceding one?

Proof. Let $U = \{f : ||f||_p \leq 1\}$. In the equal mass case, the figure is a square if and only if $p = 1$ or $\infty$ and is a circle if and only if $p = 2$. The convexity part is also standard. For non-equal mass case, it is not a square nor a circle for any $p$ (by comparing the side length or examining the candidate for "center of circle"). However, for $p = \infty$, $U$ is a square.

2. Triangle inequality.

3. If $1 < p < \infty$, prove that the unit ball in $L^p(\mu)$ is strictly convex. Show that this fails in every $L^1(\mu)$, in every $L^\infty(\mu)$, and in every $C(X)$. (Ignore trivialities, such as spaces consisting of only one point.)

Remark 1. Actually, one can prove $L^p$ is uniformly convex by Clarkson’s inequality. See [4] Problem 1.6] and [1] Section 4.3]. Moreover, Milman-Pettis Theorem states every uniformly convex space is reflexive. See [1] Section 3.7].

Proof. For $1 < p < \infty$. Given $f, g \in L^p(\mu)$ with $\|f\|_p = \|g\|_p = 1$, $f \neq g$. Minkowski inequality implies $h = \frac{1}{2}(f + g)$ has $L^p$ norm less than or equal to 1. Moreover, the equality holds if and

*Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw
only if there is \( \lambda \geq 0 \) such that \( f = \lambda g \), which is equivalent to \( f = g \) since both functions have the same \( L^p \) norm. So \( \| h \|_p < 1 \).

For \( L^1(\mu) \) and \( L^\infty(\mu) \). Suppose \( 0 < \mu(A) < \infty \) and \( 0 < \mu(B) < \infty \), and \( A \cap B = \emptyset \). Let \( f(x) = 1/\mu(A) \) when \( x \in A \) and \( f(x) = 0 \) when \( x \notin A \). Let \( g(x) = 1/\mu(B) \) when \( x \in B \) and \( g(x) = 0 \) when \( x \notin B \). Then \( f, g \) are linearly independent, and \( \| f + g \|_1 = \| f \|_1 = \| g \|_1 = 1 \).

Similarly, let \( F(x) = 1 \) when \( x \in A \) and \( F(x) = 0 \) when \( x \notin A \). Let \( g(x) = 1 \) when \( x \in A \cup B \) and \( g(x) = 0 \) when \( x \notin A \cup B \). Then \( f, g \) are linearly independent and \( \| F + g \|_\infty = \| F \|_\infty = \| g \|_\infty = 1 \).

For \( C(X) \), suppose \( X \) is a compact metric space with more than one point. (Note that if \( X \) is not compact, it’s not clear that \( \| \cdot \|_\infty \) is a norm on \( C(X) \).) Consider \( f(x) = 1/(1 + d(x, a)) \), \( g(x) = 1/(1 + d(x, a))^2 \). Then \( \| f + g \|_\infty = \| f \|_\infty = \| g \|_\infty = 1 \).

4. Let \( C \) be the space of all continuous functions on \([0, 1]\) with the supreme norm. Let \( M \) consists of all \( f \in C \) for which
\[
\int_0^{\frac{1}{2}} f(t) \, dt - \int_{\frac{1}{2}}^1 f(t) \, dt = 1.
\]

Prove that \( M \) is a closed convex subset of \( C \) which contains no elements of minimal norm.

**Remark 2.** This is related to Theorem 4.10 and Exercise 4.11 and 5.5.

**Proof.** Convexity of \( M \) is trivial. Closedness of \( M \) is through theorem on uniform convergence and integration. Next, we try to calculate the infimum. By triangle inequality, we see

\[
1 \leq \inf \{ \| f \|_\infty : f \in M \}.
\]

On the other hand, for \( n \geq 2 \), define \( f_n : [0, 1] \to \mathbb{R} \) by

\[
f_n(x) = \begin{cases} 
1 & \text{if } x \in [0, \frac{1}{2}], \\
\frac{1+n}{1-n} & \text{if } x \in [\frac{1}{2} + \frac{1}{2n}, 1], \\
\text{linear} & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]. 
\end{cases}
\]

Then \( f_n \in M \) and \( \| f_n \|_\infty = \frac{n+1}{n-1} \to 1 \) as \( n \to \infty \). So

\[
1 = \inf \{ \| f \|_\infty : f \in M \}.
\]

If there is \( f \in C \) such that \( \| f \|_\infty = 1 = \int_0^{\frac{1}{2}} f(t) \, dt - \int_{\frac{1}{2}}^1 f(s) \, ds \), then \( f(t) = 1 \) on \([0, 1/2]\) and \( f(t) = -1 \) on \((1/2, 1]\). But this contradicts to the continuity of \( f \). \( \square \)
5. Let \( M \) be the set of all \( f \in L^1([0,1]) \), relative to Lebesgue measure, such that
\[
\int_0^1 f(t) \, dt = 1.
\]
Show that \( M \) is a closed convex subset of \( L^1([0,1]) \) which contains infinitely many elements of minimal norm. (Compare this and Exercise 4 with Theorem 4.10.)

\begin{proof}
Convexity of \( M \) is trivial. Closedness of \( M \) is through triangle inequality. Note that \( \inf\{\|f\|_1 : f \in M\} \geq 1 \) and for each \( n \in \mathbb{N} \), \( f_n = n\chi_{(0,\frac{1}{n})} \in M \), distinct from other \( f_m \) and has norm 1.
\end{proof}

6. Proof. Using the continuity of \( f \), we can extend it to \( \overline{M} \) and preserve its norm.

Since \( H = \overline{M} \bigoplus M^\perp \), we extend \( f \) to \( H \) by define \( F(x) = F(x^M + x^M^\perp) := f(x^M) \). Then it’s easy to check all the assertions on \( F \) are satisfied. The uniqueness is easy to prove.

7. Construct a bounded linear functional on some subspace of some \( L^1(\mu) \) which has two (hence infinitely many) distinct norm-preserving linear extensions to \( L^1(\mu) \).

\begin{remark}
In contrast to Exercise 6, this exercise shows that such unique extension result is not true for every Banach space. Hence, no uniqueness assertion in Hahn-Banach Theorem.
\end{remark}

\begin{proof}
Consider \( L^1 = L^1([-1,1],m) \) and \( M := \{ f \in L^1 : f \equiv 0 \text{ on } [-1,0] \} \subset L^1 \). Consider the functional \( T(f) = \int_{-1}^1 f \, dx \) on \( M \) which is linear, bounded with norm 1. Such \( T \) has two distinct norm-preserving extensions, one is \( T_1(f) = \int_{-1}^1 f \, dx \), another one is \( T_2(f) = \int_{-1}^1 f\chi_{(-\frac{1}{2},1)} \, dx \).
\end{proof}

8. Proof. (a) Standard result. It’s true even for the range \( \mathbb{R} \) or \( \mathbb{C} \) is replaced by a Banach space.

(b) It’s easy to see the norm \( \leq \|x\| \). To see it’s actually an equality, we use Hahn-Banach Theorem (more precisely, Theorem 5.20).

(c) By (a)(b) and Banach-Steinhaus Theorem.

9. Proof. (a) It’s easy to see \( \|\Lambda y\| \leq \|y\|_1 \). To get the reverse inequality, we let \( x_n = (\xi_i)_{i=1}^n \), where \( \xi_i = \frac{\eta_i}{|\eta_i|} \) if \( \eta_i \neq 0 \) and \( \xi_i = 0 \) if \( \eta_i = 0 \). Then \( x_n \in c_0, \|x_n\|_\infty \leq 1 \) and \( \|\Lambda y\| \geq |\Lambda y x_n| = \sum_{i=1}^n |\eta_i| \) for every \( n \). Therefore, \( \|y\|_1 \leq \|\Lambda y\| \).

Now we prove the linear map \( y \mapsto \Lambda y \) is surjective. Given \( \Lambda \in (c_0)^* \), we let \( \eta_i = \Lambda e_i \) for each \( i \).
Let \( x_n = (\xi_i)_{i=1}^n \), where \( \xi_i = \frac{\eta_i}{|\eta_i|} \) if \( \eta_i \neq 0 \) and \( \xi_i = 0 \) if \( \eta_i = 0 \). Then \( x_n \in c_0, \|x_n\|_\infty \leq 1 \) and
\[ \|\Lambda\| \geq |\Lambda x_n| = \sum_{i=1}^{n} |\eta_i| \text{ for every } n. \] Therefore, \[ \sum_{i=1}^{\infty} |\eta_i| \leq \|\Lambda\|, \] that is, \( y = (\eta_i) \in l^1 \). Given \( x = (\xi_i)_{i=1}^{\infty} \in c_0 \), we see \( x_n = (\xi_i)_{i=1}^{n} \to x \) in \( l^\infty \). By the continuity of \( \Lambda \),

\[
\Lambda x = \lim_{n \to \infty} \Lambda x_n = \lim_{n \to \infty} \sum_{i=1}^{n} \xi_i \Lambda e_i = \lim_{n \to \infty} \sum_{i=1}^{n} \xi_i \eta_i = \Lambda y x.
\]

(b) It’s easy to see \( \|\Lambda y\| \leq \|y\|_{\infty} \). To get the reverse inequality, we may assume \( \|y\|_{\infty} > 0 \), then for each small \( \epsilon > 0 \), there is some \( |\xi_i| > \|y\|_{\infty} - \epsilon \). Take \( x = e_i \), then \( \|x\|_1 = 1 \) and hence \( \|\Lambda\| \geq |\Lambda x| = |\xi_i| > \|y\|_{\infty} - \epsilon \). Letting \( \epsilon \to 0 \), we see \( \|\Lambda\| \geq \|y\|_{\infty} \).

Now we prove the linear map \( y \mapsto \Lambda y \) is surjective. Given \( \Lambda \in (l^1)^* \), let \( \eta_i = \Lambda e_i \) and hence \( |\eta_i| \leq \|\Lambda\| \) for each \( i \), that is, \( y = (\eta_i) \in l^1 \). Given \( x = (\xi_i)_{i=1}^{\infty} \in l^1 \), we see \( x_n = (\xi_i)_{i=1}^{n} \to x \) in \( l^1 \). By the continuity of \( \Lambda \),

\[
\Lambda x = \lim_{n \to \infty} \Lambda x_n = \lim_{n \to \infty} \sum_{i=1}^{n} \xi_i \Lambda e_i = \lim_{n \to \infty} \sum_{i=1}^{n} \xi_i \eta_i = \Lambda y x.
\]

(c) By Hahn-Banach Theorem (Theorem 5.20) and (a).

(d) Consider the collection \( S \) of all elements \( (x_i)_{i=1}^{\infty}, x_i = 1 \) or 0 for each \( i \) which is an uncountable subset of \( l^\infty \) and for each \( x, y \in S, x \neq y, \|x - y\|_{\infty} = 1 \). So every dense subset of \( l^\infty \) is uncountable.

Let \( S \) be the collection of all elements \( (x_i)_{i=1}^{\infty}, x_i \in \mathbb{Q} \) for each \( i \) and \( x_i = 0 \) for all \( i \geq N \) for some \( N \). By understanding the collection \( S \) is countable and dense in \( l^1 \) and \( c_0 \), we see \( l^1 \) and \( c_0 \) are separable.

10. If \( \sum \alpha_i \xi_i \) converges for every sequence \( \{\xi_i\} \) such that \( \xi_i \to 0 \) as \( i \to \infty \), prove that \( \sum |\alpha_i| < \infty \).

**Remark 4.** The assumption is weaker than Exercise 6.4 and Folland [2, Theorem 6.14].

**Proof.** For each \( n \in \mathbb{N} \), define \( \Lambda_n : c_0 \subset l^\infty \to \mathbb{C} \) by

\[
\Lambda_n(\xi) = \sum_{i=1}^{n} \alpha_i \xi_i.
\]

It’s easy to check \( c_0 \) with sup-norm is a Banach space, each \( \Lambda_n \) is linear, bounded with norm \( \|\Lambda_n\| = \sum_{i=1}^{n} |\alpha_i| \) and \( \{\Lambda_n\} \) is pointwisely bounded. By Banach-Steinhaus Theorem, we see there is \( M > 0 \) such that \( M \geq \|\Lambda_n\| = \sum_{i=1}^{n} |\alpha_i| \) for all \( n \). Therefore, \( \sum_{i=1}^{\infty} |\alpha_i| \) exists (due to monotonicity and boundedness) and \( M \geq \sum_{i=1}^{\infty} |\alpha_i| \).

11. Let \( \beta \in (0, 1) \). Prove that \( C^{0,\beta}([a, b]; \mathbb{C}) \) are Banach spaces with norms \( \|f\|_1 = |f(a)| + [f]_{0,\beta} \) and \( \|f\|_2 = \|f\|_{\infty} + [f]_{0,\beta} \), where \( [f]_{0,\beta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta} \).
Proof. Let $U = [a, b]$. For simplification, we only consider the completeness with respect to $\| \cdot \|_2$. Given a Cauchy sequence $\{f_n\} \subset C^{0,\beta}(\overline{U})$, then there is a $f \in C(\overline{U})$ such that $f_n \to f$ uniformly and a constant $M > 0$ such that for each $n \in \mathbb{N}$ and $s \neq t$,

$$\frac{|f_n(s) - f_n(t)|}{|s - t|^{\beta}} \leq M.$$

Then for each $s \neq t$, there is some $N = N(s, t) \in \mathbb{N}$ such that

$$|f(s) - f(t)| \leq |f(s) - f_N(s)| + |f_N(s) - f_N(t)| + |f_N(t) - f(t)|$$

$$\leq 2|s - t|^\beta + M|s - t|^\beta.$$

Therefore, $f \in C^{0,\beta}(\overline{U})$. It remains to show $f_n \to f$ in $C^{0,\beta}(\overline{U})$. For each $s, t \in U$ and $\epsilon > 0$, by Cauchy’s criteria, we can find a $N = N(\epsilon)$ such that for $k, n \geq N$

$$|f_k(s) - f_n(s) - f_k(t) + f_n(t)| \leq \epsilon|s - t|^\beta.$$

For each $\eta > 0$, we can find a $K = K(\eta, s, t) > N$, such that $|f(x) - f_K(x)| \leq \eta$ for $x = s$, or $x = t$. Therefore, for every $n \geq N$

$$|f(s) - f_n(s) - f(t) + f_n(t)| \leq |f(s) - f_K(s)| + |f_K(s) - f_n(s) - f_k(t) + f_n(t)| + |f(t) - f_K(t)| \leq 2\eta + \epsilon|s - t|^\beta.$$

Letting $\eta \to 0$, we obtain that $|f(s) - f_n(s) - f(t) + f_n(t)| \leq \epsilon|s - t|^\beta$ for all $n \geq N(\epsilon)$.

**Remark** 5. It is true for the general case $C^{k,\beta}(U)$, $U$ is connected in $\mathbb{R}^d$, with an almost identical proof as the above case $k = 0$, except we need the standard convergence theorem between $\{f_n\}$ and $\{Df_n\}$.

12. **Remark** 6. See Notes and Comments for Section 5.22.

**Proof.**

13. **Proof.** (a) By the pointwise convergence assumption,

$$X = \bigcup_M E_M := \bigcup_M \cap_n \{x : |f_n(x)| \leq M\}.$$

Note that $E_M \subset E_{M+1}$ is nonempty for large $M$, and each $E_M$ is closed since it’s intersection of closed sets. Therefore, by Baire Category Theorem, there is some $M$ such that $E_M$ contains an open set $V \neq \emptyset$. (b)’s proof is similar to (a) (see Hint.)

14. **Proof.**
15. **Proof.**

16. **Remark 7.** See Rudin [3, Theorem 2.11, 2.14-15 and Theorem 1.24].

**Theorem 8** (Closed Graph Theorem). Let $L$ be a linear map between two $F$-spaces (complete translation-invariant metric vector spaces), then the graph of $L$ is closed iff $L$ is continuous.

The corresponding open mapping theorem has a setting in $F$-spaces.

17. **Proof.** (a) It’s easy to see $\|f\|_\infty$ dominates the norm of multiplication operator $M_f(\cdot) = f$.

(b)

18. **Proof.** Given $x \in X$, for each $\epsilon > 0$, there exists $y = y(x, \epsilon) \in E$ such that $\|y - x\|_X < \frac{\epsilon}{M}$, then

$$\|\Lambda_n x - \Lambda_m x\|_Y \leq \|\Lambda_n x - \Lambda_n y\|_Y + \|\Lambda_n y - \Lambda_m y\|_Y + \|\Lambda_m y - \Lambda m y\|_Y \leq 2M \frac{\epsilon}{M} + \|\Lambda_n y - \Lambda_m y\|_Y$$

The last term is less than $\epsilon$ if $n, m > N$ for some $N = N(y) = N(x, \epsilon)$. So $\{\Lambda_n x\}$ is Cauchy in the Banach space $Y$.

19. **Proof.** (a) We try to apply Exercise 18 with $X = Y = C(T), \Lambda_n f = \frac{s_n(f)}{\log n}$. Then we see $\Lambda_n$ converges to 0 pointwisely on the dense set $E = P(T)$, the set of all trigonometric polynomials.

It remains to show that $\Lambda_n$ is uniformly bounded which is due to

$$(\log n)\|\Lambda_n\| = \int_0^\pi \frac{\sin((n + \frac{1}{2})x)}{\sin x/2} dx \leq \frac{2}{1 - \frac{\pi^2}{12}} \int_0^\pi \frac{\sin((n + \frac{1}{2})x)}{x} dx \leq C \sum_{k=1}^n \frac{1}{k \pi}.$$  

(b) Use the Banach-Steinhaus Theorem in exactly the same way as Section 5.11 with the change of the linear functionals to $\Lambda_n f = \frac{1}{\lambda_n} s_n(f; 0)$. Note that $\|\Lambda_n\| = \|D_n\|_1/\lambda_n$. Note that

$$\frac{\|D_n\|_1}{\lambda_n} \geq \frac{4}{\pi^2|\lambda_n|} \sum_{k=1}^n \frac{1}{k} \geq \frac{4}{\pi^2|\lambda_n|} (\ln(n) + \gamma) = \frac{4}{\pi^2} \frac{\ln(n)}{|\lambda_n|} + \frac{\gamma}{|\lambda_n|} \to \infty,$$

since you are given $\frac{\lambda_n}{\log(n)} \to 0$. Here $\gamma$ is the Euler Mascheroni constant.

So actually, it’s unbounded for all $f$ in some dense $G_\delta$ set in $C(T)$.

20. **Proof.** (a) No, since $\mathbb{Q}$ is not a $G_\delta$ set, but the set of points $A$ at which a sequence of positive continuous functions is unbounded is $\cap_m \cup_n \{x : f_n(x) > m\}$ which is a $G_\delta$ set. (If $\mathbb{Q} = \cap_n V_n$, $V_n$ open, then $\mathbb{R} = \{r_m\} \cup (\cup_n V_n^c)$ which is of first category. A contradiction!)

(b) Let $\mathbb{Q} = \{q_k\}$, we consider

$$f_n(x) = \min_{1 \leq k \leq n} \{k + n|x - q_k|\} \geq 1.$$
Then for each $q_m \in \mathbb{Q}$, we see for $n \geq m$, $f_n(q_m) \leq m + n|q_m - q_m| = m$, so $\{f_n(q_m)\}$ is bounded. On the other hand, if $x \in \mathbb{Q}^c$, then given $M > 0$, there is some $N = N(M) > M$ such that $n|x - q_i| > M$ for all $1 \leq i \leq M$ provided $n > N$. Therefore,

$$f_n(x) = \min_{1 \leq k \leq n} \{k + n|x - q_k|\} > M,$$

that is, $f_n(x) \to \infty$.

(c) Irrational part is answered in (b). The answer is affirmative for rational part:

Let $A_n = \cup_{i=1}^n (q_i - \epsilon_n, q_i + \epsilon_n)$, where $0 < \epsilon_n = \frac{1}{4} \min\{|q_i - q_j|: 1 \leq i < j \leq n + 1\} \setminus 0$.

Take $f_n$ to be the zig-zag continuous function that equals to $n$ at $q_1, \ldots, q_n$ and 0 outside $A_n$. Then for every $x \in \mathbb{Q}$, $f_n(x) = n \to \infty$. On the other hand, given $x \in \mathbb{Q}^c$, suppose for some $n_0 \in \mathbb{N}$, $x \in A_n$ for all $n \geq n_0$. Then for some fixed $1 \leq i \leq n_0$, $x \in (q_i - \epsilon_{n_0}, q_i + \epsilon_{n_0})$. Since $x \in A_{n_0+1}$, by construction of $\epsilon_n$, $x \in (q_i - \epsilon_{n_0+1}, q_i + \epsilon_{n_0+1})$. Inductively, we know $x \in (q_i - \epsilon_n, q_i + \epsilon_n)$ for all $n \geq n_0$, and hence $x = q_i$ since $\epsilon_n \to 0$. This contradicts to $x \in \mathbb{Q}^c$.

So for every $M \in \mathbb{N}$, there is some $m > M$ such that $x \not\in A_m$, that is, $f_m(x) = 0$, and therefore $f_n(x) \not\to \infty$. 


22. Proof.

References


