3.1 Fourier Coefficients

1. Proof.

2. Proof.

3. Proof.

4. Proof.

5. Proof.

6. Proof.

7. Proof.

3.2 Reproduction of Functions from Their Fourier Coefficients

1. On $\mathbb{T}^1$ let $P$ be a trigonometric polynomial of degree $N > 0$. Show that $P$ has at most $2N$ zeros. Construct a trigonometric polynomial with exactly $2N$ zeros.

Proof.
2. (Hausdorff-Young inequality) Prove that when \( f \in L^p(T^n), 1 \leq p \leq 2 \), the sequence of Fourier coefficients of \( f \) is in \( l^{p'}(\mathbb{Z}^n) \) and

\[
\left( \sum_{m \in \mathbb{Z}^n} |\hat{f}(m)|^{p'} \right)^{1/p'} \leq \|f\|_{L^p(T^n)}.
\]

Also observe that 1 is the best constant in the preceding inequality.

**Proof.**

3. Use without proof that there exists a constant \( C > 0 \) such that

\[
\sup_t \left| \sum_{k=2}^N e^{ik \log k} e^{ikt} \right| \leq C \sqrt{N}, \quad N = 2, 3, 4, \ldots
\]

to construct an example of a continuous function \( g \) on \( T^1 \) with

\[
\sum_{m \in \mathbb{Z}} |\hat{g}(m)|^q = \infty
\]

for all \( q < 2 \). Thus the Hausdorff-Young inequality of Exercise 3.2.2 fails for \( p > 2 \).

**Hint:** Consider \( g(x) = \sum_{k=2}^{\infty} e^{ik \log k} e^{2\pi i k x} \). For a proof of the previous estimate, see Zygmund’s Trigonometric Series Vol. I, Theorem (4.7) p. 199.

**Proof.**

**Remark** 1. Also see Exercise 3.3.8.

4. (S. Bernstein) Let \( P(x) \) be a trigonometric polynomial of degree \( N \) on \( T^1 \).

Prove that \( \|P'\|_{L^\infty} \leq 4\pi N \|P\|_{L^\infty} \).

**Hint:** Prove first that \( P'(x)/2\pi i N \) is equal to

\[
((e^{-2\pi i N(x)} P) \ast F_{N-1})(x)e^{2\pi i N x} - ((e^{2\pi i N(x)} P) \ast F_{N-1})(x)e^{-2\pi i N x}
\]

and then take \( L^\infty \) norms.

**Proof.**

5. (Fejér and F. Riesz) Let \( P(\xi) = \sum_{k=0}^N a_k e^{2\pi i k \xi} \) be a trigonometric polynomial on \( T^1 \) of degree \( N \) such that \( P(\xi) > 0 \) for all \( \xi \). Prove that there exists a trigonometric polynomial \( Q(\xi) \) of the form \( \sum_{k=0}^N b_k e^{2\pi i k \xi} \) such that \( P(\xi) = |Q(\xi)|^2 \).

**Hint:** Since \( P \geq 0 \) the complex-variable polynomial \( R(z) = \sum_{k=0}^N a_k z^{k+N} \) must satisfy \( R(z) = z^{2N} R(1/z) \), and thus it must have \( N \) zeros inside the unit circle and the other \( N \) outside. Therefore we may write \( R(z) = a_N \prod_{k=1}^s (z z_k)^{r_k} (z 1/\overline{z})^{r_k} \) for some \( 0 < |z_k| < 1 \) and \( r_k \geq 1 \) with \( \sum_{k=1}^s r_k = N \). Then take \( z = e^{2\pi i \xi} \).
6. Let \( g \) be a function on \( \mathbb{R}^n \) that satisfies \( |g(x)| + |\hat{g}(x)| \leq C(1 + |x|)^{n\delta} \) for some \( C, \delta > 0 \) and all \( x \in \mathbb{R}^n \). Prove that
\[
\lambda^n \sum_{m \in \mathbb{Z}^n} \hat{g}(\lambda m + \alpha) e^{2\pi i x \cdot (m + \alpha)} = \sum_{k \in \mathbb{Z}^n} g\left(\frac{x + k}{\lambda}\right) e^{2\pi i k \alpha}
\]
for any \( x, \alpha \in \mathbb{R}^n \) and \( \lambda > 0 \).

Proof.

7. Verify the following identity when \( 0 < r < 1 \) and \( x \in \mathbb{R}^n \)
\[
\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \sum_{k \in \mathbb{Z}^n} \frac{1}{2\pi} \log \frac{1}{\pi r} \left( \frac{1}{2\pi} \log \frac{1}{\pi r} \right)^2 + |x - k|^2 = \sum_{m \in \mathbb{Z}^n} r^{|m|} e^{2\pi i m x}.
\]
In the special case \( n = 1 \) and \( x \in \mathbb{R} \) we have
\[
\frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \log \frac{1}{\pi r} \left( \frac{1}{2\pi} \log \frac{1}{\pi r} \right)^2 + |x - k|^2 = \frac{1 - r^2}{1 - 2 r \cos(2\pi x) + r^2}.
\]

Proof.

8. Let \( \gamma \in \mathbb{R} \) and \( \lambda > 0 \). Show that
\[
\sum_{k \in \mathbb{Z}} \frac{\cos(2\pi k \gamma)}{\lambda^2 + k^2} = \frac{\pi}{\lambda} \frac{\cosh(2\pi \gamma - [\gamma] - \frac{1}{2})}{\sinh(\pi \lambda)}.
\]

Hint: Use Exercise 3.2.6 \((n = 1)\) with \( x = 0, \lambda = -\gamma \lambda, g(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \) and sum in \( m \).

Proof.

### 3.3 Decay of Fourier Coefficients

1. Given a sequence \( \{a_n\}_{n=0}^{\infty} \) of positive numbers such that \( a_n \to 0 \) as \( n \to \infty \), find a nonnegative integrable function \( h \) on \([0, 1]\) such that
\[
\int_0^1 h(t) t^m \, dt \geq a_m.
\]
Use this result to deduce a different proof of Lemma 3.3.2.

[Hint: Try \( h = e \sum_{k=0}^{\infty} (\sup_{j \geq k} a_j - \sup_{j \geq k+1} a_j)(k + 2) \chi_{[k+1, k+2]} \).]
2. Prove that given a positive sequence \( \{d_m\}_{m \in \mathbb{Z}^n} \) with \( d_m \to 0 \) as \( |m| \to \infty \), there exists a positive sequence \( \{a_j\}_{j \in \mathbb{Z}} \) with \( a_{m_1} \cdots a_{m_n} \geq d_{(m_1, \ldots, m_n)} \) and \( a_j \to 0 \) as \( |j| \to \infty \).

Proof. For simplicity, we assume \( n = 2 \). Let \( A_i := \max_{n \in \mathbb{Z}} \sqrt{|d_{i,n}|} \) and \( B_j = \max_{m \in \mathbb{Z}} \sqrt{|d_{m,j}|} \), where the maximum exists for each \( i, j \in \mathbb{Z} \) by the decay assumption. Let \( a_i = \max\{A_i, B_i\} \). Then for each \( (m_1, m_2) \in \mathbb{Z}^2 \), we have \( a_{m_1}a_{m_2} \geq A_{m_1}B_{m_2} \geq \sqrt{|d_{m_1,m_2}|} = d_{m_1,m_2}. \)

3. (a) Use the idea of the proof of Lemma 3.3.3 to prove that if a twice continuously differentiable function \( f \geq 0 \) is defined on \((0, \infty)\) and satisfies \( f'(x) \leq 0 \) and \( f''(x) \geq 0 \) for all \( x > 0 \), then \( \lim_{x \to \infty} x f'(x) = 0 \).

(b) Suppose that a continuously differentiable function \( g \) is defined on \((0, \infty)\) and satisfies \( g \geq 0, g' \leq 0 \), and \( \int_1^\infty g(x) \, dx < \infty \). Prove that \( \lim_{x \to \infty} x g(x) = 0 \).

Proof. (a) By hypothesis, \( 0 \geq \frac{\pi}{2} f'(x) \geq \int_x^\infty f'(t) \, dt = f(x) - f\left(\frac{\pi}{2}\right) \to 0 \) as \( x \to \infty \).

(b) By hypothesis, \( 0 \leq \frac{\pi}{2} g(x) \leq \int_x^\infty g(t) \, dt \to 0 \) as \( x \to \infty \).

4. It’s easy to show that for \( 0 < \gamma < \delta < 1 \), we have \( \|f\|_{\Lambda_\gamma} \leq m(T^n)^{\delta-\gamma} \|f\|_{\Lambda_\delta} \) for all functions \( f \), where \( m \) is for Lebesgue measure, and thus \( \Lambda_\delta \) is a subspace of \( \Lambda_{\gamma} \).

5. Suppose that \( f \) is a differentiable function on \( T^1 \) whose derivative \( f' \) is in \( L^2(T^1) \). Prove that \( f \in A(T^1) \) and that

\[
\|f\|_{A(T^1)} \leq \|f\|_{L^1} + \frac{1}{2\pi} \left( \sum_{j \neq 0} j^{-2} \right)^{1/2} \|f'\|_{L^2}.
\]

Proof. One note that

\[
\sum_{m \in \mathbb{Z}} |\hat{f}(m)| = |\hat{f}(0)| + \sum_{m \in \mathbb{Z} \setminus \{0\}} |2\pi im\hat{f}(m)/(2\pi|m|)| = \|f\|_{L^1} + \frac{1}{2\pi} \sum_{m \neq 0} \frac{\hat{f}(m)}{|m|}.
\]

Apply the Cauchy-Schwarz’s inequality and Planchel’s theorem to complete the proof.

6. (a) Prove that the product of two functions in \( A(T^n) \) is also in \( A(T^n) \) and that

\[
\|fg\|_{A(T^n)} \leq \|f\|_{A(T^n)} \|g\|_{A(T^n)}.
\]

(b) Prove that the convolution of two square integrable functions on \( T^n \) always gives a function in \( A(T^n) \).
Proof. (a) By Fubini-Tonelli’s theorem
\[
\sum_{m} |\hat{f}(m)| = \sum_{m} \left| \sum_{k} \hat{f}(k)\hat{g}(m-k) \right| \leq \sum_{m} \left| \sum_{k} \hat{f}(k)\hat{g}(m-k) \right| = \|f\|_{\Lambda(T^n)} \|g\|_{\Lambda(T^n)}
\]
(b) is a simple consequence of Cauchy-Schwarz’s inequality.

7. Fix \(0 < \alpha < 1\) and define \(f\) on \(T^1\) by setting \(f(x) = \sum_{k=0}^{\infty} 2^{-\alpha k} e^{2\pi i 2^k x}\).

Prove that the function \(f\) lies in \(\dot{\Lambda}_\alpha(T^1)\). Conclude that there does not exist positive \(\beta > \alpha\) such that for all \(f\) in \(\dot{\Lambda}_\alpha(T^1)\) we have \(\sup_{m \in \mathbb{Z}} |m|^\beta |\hat{f}(m)| < \infty\).

Proof. The second assertion is easy, we omit. We prove the first one by following the hint.

Given \(x, h \in T^1, h \neq 0\). Pick \(N \in \mathbb{N}\) such that \(2^N|h| > 1 \geq 2^{N-1}|h|\). Then
\[
|f(x+h) - f(x)| \leq \left| \sum_{k=0}^{N} 2^{-\alpha k} e^{2\pi i 2^k x}[e^{2\pi i 2^k h} - 1] \right| + \left| \sum_{k=N}^{\infty} 2^{-\alpha k} e^{2\pi i 2^k x}[e^{2\pi i 2^k h} - 1] \right|
\]
\[
\leq \sum_{k=0}^{N} 2^{1-\alpha} |h| + 2 \sum_{k=N}^{\infty} 2^{-\alpha k} \leq \frac{2^{(1-\alpha)(N+1)}}{2^{1-\alpha} - 1} |h| + \frac{2}{1 - 2^{-\alpha}} 2^{-\alpha(N+1)}
\]
\[
\leq \left( \frac{4^{1-\alpha}}{2^{1-\alpha} - 1} + \frac{21-\alpha}{1 - 2^{-\alpha}} \right) |h|^\alpha
\]

8. As same reference as Exercise 3.2.3, there exists a constant \(C > 0\) such that
\[
\sup_{t \in \mathbb{R}} \left| \sum_{k=2}^{N} e^{i k \log k} e^{ikt} \right| \leq C \sqrt{N}, \quad N = 2, 3, 4, \ldots
\]

We can use this estimate to prove that the function
\[
f(x) = \sum_{k=2}^{\infty} \frac{e^{i k \log k}}{k} e^{2\pi i kx}
\]
is in \(\dot{\Lambda}_{\frac{1}{2}}(T^1)\) but not in \(A(T^1)\). Conclude that the restriction \(s > \frac{1}{2}\) in Theorem 3.3.16 is sharp.
Proof. It’s clear that \( f \not\in A(T) \). Using the summation by parts one deduce the \( N \)-th partial sum of \( f \) as (where \( B_j = \sum_1^j b_n \))

\[
f_N(x) = \frac{1}{N} B_N(x) + \sum_{j=1}^{N-1} \frac{1}{j(j+1)} B_j(x).
\]

Then \( f_N \) converges absolutely by the given fact and hence to \( f \) uniformly. Also, letting \( N \to \infty \) we obtain

\[
f(x + h) - f(x) = \sum_{j=1}^{\infty} \left( B_j(x + h) - B_j(x) \right) \frac{1}{j(j+1)} = \sum_{j=1}^{N} + \sum_{j=N+1}^{\infty} := P + Q.
\]

Let \( 0 < h < 1, N = \left\lceil \frac{1}{h} \right\rceil \). The terms of \( |Q| \) are \( O(\sqrt{j})j^{-2} \) so that

\[
|Q| = O(N^{-\frac{3}{2}}) = O(h^{\frac{3}{2}}).
\]

On the other hand, we apply the summation by parts again to see

\[
B'_j(z) = \sum_{k=1}^j ke^{ik\log k} e^{ikz} = jB_j(z) + \sum_{k=1}^{j-1} B_k(z) = O(j^{3/2}).
\]

Therefore, applying the mean value theorem to the real and imaginary parts of \( B_j(x+h) - B_j(x) \), we get

\[
|P| \leq \sum_{j=1}^{N} O(h^{3/2})j^{-2} = O(h^{1/2}N^{1/2}) = O(h^{\frac{1}{2}}).
\]

Therefore \( |f(x + h) - f(x)| \leq O(h^{\frac{1}{2}}) \) for each \( 0 < h < 1 \), that is, \( f \in C^{1/2}(T) \).

\[\square\]

**Remark 3.** In Bernstein’s theorem the pointwise Hölder condition can be relaxed to \( L^2 \) Hölder condition.

**Remark 4.** Another classical example to explain the optimality is the Rudin-Shapiro’s polynomial. Note that the proof needs the Bernstein’s inequality stated in Exercise 3.2.4. See [4, Section 1.4.6] and [2, page 99-100].

**Remark 5.** However, Zygmund prove that under the additional hypothesis \( f \) is of bounded variation, the Bernstein’s theorem is true for all \( \alpha > 0 \). See [3, Page 44].

9. It’s standard to show that the Fourier transform from \( L^1(T^n) \) to \( C_0(Z^n) \) is not surjective, that is, there exist sequences \( \{a_m\}_{m \in Z^n} \) that tend to zero as \( |m| \to \infty \) for which there do not exist functions \( f \) in \( L^1(T^n) \) with \( \widehat{f}(m) = a_m \) for all \( m \).
3.4 Pointwise Convergence of Fourier Series

1. We omit the routine computations for Fourier coefficients.

2. It’s easy to use Exercise 3.4.1 and Proposition 3.4.2 to prove that

\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k+1)^2} = \frac{\pi^2}{4}, \quad \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^k}{k^2+1} = \frac{2\pi}{e^\pi - e^{-\pi}}.
\]

3. Let \( M > N \) be given positive integers.

(a) For \( f \in L^1(\mathbb{T}) \), prove the following identity:

\[
(D_N \ast f)(x) = \frac{M+1}{M-N} (F_M \ast f)(x) - \frac{N+1}{M-N} (F_N \ast f)(x) - \frac{M+1}{M-N} \sum_{N < |j| \leq M} \left( 1 - \frac{|j|}{M+1} \right) \hat{f}(j) e^{2\pi ijx}.
\]

(b) (G. H. Hardy) Suppose that a function \( f \) on \( \mathbb{T} \) satisfies the following condition:

for any \( \epsilon > 0 \) there exists an \( a > 1 \) and a \( k_0 > 0 \) such that for all \( k \geq k_0 \) we have

\[
\sum_{k < |m| \leq [ak]} |\hat{f}(m)| < \epsilon.
\]

Use part (a) to prove that if \((F_N \ast f)(x)\) converges (uniformly) to \( A(x) \) as \( N \to \infty \), then \((D_N \ast f)(x)\) also converges (uniformly) to \( A(x) \) as \( N \to \infty \).

**Proof.** (a) Compare the Fourier coefficients and use the uniqueness theorem to the continuous function \( D_N \ast f \).

(b) Given \( \epsilon > 0 \), pick \( N_0 \in \mathbb{N} \) such that \( \|F_N \ast f - A\|_\infty \leq \frac{\epsilon}{100 \frac{a(k_0)}{a(\frac{5}{4})-1}} \) if \( N \geq N_0 \).

Then for each \( N > \max\{N_0, \frac{2}{a(\frac{5}{4})-1}, k_0(\frac{5}{2})\} \), we have, by (a) with \( M = [a(\frac{5}{4})]N \),

\[
\|D_N \ast f - A\|_\infty \leq \frac{a(\frac{5}{4})N+1}{a(\frac{5}{4})N-N-1} \|F_M \ast f - A\|_\infty + \frac{N+1}{a(\frac{5}{4})N-N-1} \|F_N \ast f - A\|_\infty + \sum_{N < |j| \leq [aN]} |\hat{f}(j)| \\
\leq \frac{3a(\frac{5}{4})-1}{a(\frac{5}{4})-1} \frac{\epsilon}{100 \frac{3a(\frac{5}{4})-1}{a(\frac{5}{4})-1}} + \frac{3a(\frac{5}{4})-1}{a(\frac{5}{4})-1} \frac{\epsilon}{100 \frac{3a(\frac{5}{4})-1}{a(\frac{5}{4})-1}} + \frac{\epsilon}{2} < \epsilon.
\]

\( \square \)

4. It’s easy to use Proposition 3.4.2 and Exercise 3.4.1(b) to show that for \( 0 < b < \frac{1}{2} \) we have

\[
\lim_{N \to \infty} \sum_{m=-N, m \neq 0}^{N} \frac{\sin(2\pi bm)}{m\pi} e^{2\pi ibm} = \frac{1}{2} - 2b.
\]
5. Let \( f \) be an integrable function on \( \mathbb{T}^n \) and \( g \) be a bounded function on \( \mathbb{T}^n \) and let \( K \) be a compact subset of \( \mathbb{T}^n \). Consider the family \( \mathcal{F} = \{ f_w : w \in K \} \), where \( f_w(x) = f(x - w)g(x) \) for all \( x \in \mathbb{T}^n \). Prove that the Riemann-Lebesgue lemma holds uniformly for the family \( \mathcal{F} \). This means that given \( \epsilon > 0 \) there exists an \( N_0(K) > 0 \) such that for \( |m| \geq N_0 \) we have \( |\hat{f}_w(m)| \leq \epsilon \) for all \( w \in K \).

Proof. The key observation is the following: For any \( \epsilon > 0 \), there is \( N = N(\epsilon) \in \mathbb{N} \) and \( \{ w_i \}_{i=1}^N \subset K \) such that \( \mathcal{F} \subset \bigcup_{i=1}^N \mathcal{B}(f_{w_i}, \frac{\epsilon}{10}) \), where the distance is measured by \( L^1 \) norm. To prove this, we note that for each \( w, v \in K \),

\[
\int_{\mathbb{T}^n} |f_w(x) - f_v(x)| \, dx = \int_{\mathbb{T}^n} |f(x - w) - f(x - v)||g(x)| \, dx \leq \|g\|_\infty \int_{\mathbb{T}^n} |f(x - (w - v)) - f(x)| \, dx
\]

then the above assertion follows from the \( L^1 \)-continuity of translation and compactness of \( K \).

The desired result is now reduced to the case that \( K \) has finite points and becomes trivial. \( \square \)

6. By Exercise 3.4.5 and the proof of (Dini-Tonelli’s) Theorem 3.4.7, we have the following version of Corollary 3.4.8 (b):

Suppose that a function \( f \) on \( \mathbb{T}^n \) is constant on the cross \( U = \{(x_1, \cdots, x_n) \in \mathbb{T}^n : |x_j - a_j| < \delta \text{ for some } j\} \), for some \( \delta < 1/2 \). Then \( D_N^n * f \) converges to \( f(a) \) uniformly on compact subsets of the box \( W = \{(x_1, \cdots, x_n) \in \mathbb{T}^n : |x_j - a_j| < \delta \text{ for all } j\} \).

7. To obtain a constructive proof of the existence of a continuous function whose Fourier series diverges at a point. We refer reader to Stein-Shakarchi’s Book I, *Fourier Analysis*, Section 3.2.

### 3.5 A Tauberian theorem and Functions of Bounded Variation

1. Proof.
2. Proof.
3. Proof.
4. Proof.
5. Proof.
6. Proof.
3.6 Lacunary Series and Sidon Sets

1. Proof.

2. Proof.

3. Proof.

4. Proof.

5. Proof.

6. Proof.

7. Proof.

8. Proof.

References


