Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 7  Sobolev Spaces*

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1. Proof. □

2. Proof. □

3. Proof. □

4. **Prove that the product formula**

\[ D(uv) = vDu + uDv \]

holds for all \( u, v \in W^1(\Omega) \) such that \( uv, uDv + vDu \in L^1_{loc}(\Omega) \).


**Proof.** Step 1 is to prove the theorem under the additional assumption \( v \in C^1 \). This is standard, so we just state it without proof.

Step 2 is to prove the theorem under the additional assumption \( u \in W^1(\Omega) \cap L^\infty(\Omega) \). This is proved by Step 1 and Theorem 7.4, a characterization of weakly differentiability. This is also standard.

Step 3 is consider the problem under the assumption \( u, v \geq 0 \): we first assume \( v \geq 1 \) and define \( U_n := \min\{u, \frac{n}{v}\} \). By Lemma 7.6, \( U_n \in W^1(\Omega) \cap L^\infty(\Omega) \), and \( D(U_nv) = Du \chi_{\{uv<n\}} + \left(-\frac{nDv}{v^2}\right) \chi_{\{uv \geq n\}} \). Hence \( U_n Dv + v DU_n = (uDv + vDu) \chi_{\{uv<n\}} \in L^1_{loc}(\Omega) \). Step 2 implies that, for each \( \phi \in C^1_0(\Omega) \),

\[ \int_\Omega U_nv D\phi = -\int_\Omega [U_n Dv + v DU_n] \phi. \]

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Note that $U_n v = \min\{uv, n\} \nearrow uv$, $U_n Dv + v D U_n \rightarrow u Dv + v Du$ a.e. and $|U_n Dv + v D U_n| \leq |u Dv + v Du| \in L^1(supp \phi)$, MCT and LDCT imply that

$$\int \Omega uv D \phi = \left( \int_{\{D \phi \geq 0\}} + \int_{\{D \phi < 0\}} \right) uv D \phi = \lim_{n \to \infty} \left( \int_{\{D \phi \geq 0\}} + \int_{\{D \phi < 0\}} \right) U_n v D \phi$$

$$= \lim_{n \to \infty} \int \Omega U_n v D \phi = \lim_{n \to \infty} \int \Omega [U_n Dv + v D U_n] \phi = - \int \Omega (u Dv + v Du) \phi.$$

The above formula is also true for general $v \geq 0$, since

$$\int \Omega uv D \phi + \int \Omega u D \phi = \int \Omega u(v+1) D \phi = - \int \Omega (u D(v+1) + (v+1) Du) \phi = - \int \Omega (u Dv + v Du) \phi - \int \Omega D u \phi.$$

Finally, we decompose $u = u^+ - u^-, v = v^+ - v^-$. Since $u^+ v^+ = uv \chi_{\{u > 0, v > 0\}}$ and $u^+ D v^+ + v^+ D u^+ = (u Dv + v Du) \chi_{\{u > 0, v > 0\}}$, they are in $L^1_{loc}(\Omega)$. Other similar functions also in $L^1_{loc}(\Omega)$ without proof. Then by Step 3, for each $\phi \in C^1_0(\Omega)$,

$$\int \Omega uv D \phi = \int \Omega (u^+ v^+ - u^+ v^- - u^- v^+ + u^- v^-) D \phi$$

$$= - \int \Omega \left( (u^+ D v^+ + v^+ D u^+) - (u^+ D v^- + v^- D u^+) - (u^- D v^+ + v^+ D u^-) + (u^- D v^- + v^- D u^-) \right) \phi$$

$$= - \int \Omega (u Dv + v Du) \phi.$$

\[ \square \]

**Remark 2.** In the same website, Mateusz Kwanicki (from Wrocław University of Science and Technology?) informed me that this problem may be solved by the characterization of absolutely continuity along any line. He said:

"How about using the ACL characterisation of differentiability? I mean, $u$ has weak partial derivatives if and only if it is absolutely continuous on almost every line in any cardinal direction, and the classical partial derivatives (defined thus almost everywhere) are integrable.

Since the product of absolutely continuous functions is absolutely continuous, with $(f g)' = f'g + fg'$, it follows that $uv$ has the ACL property, and the classical partial derivatives of $uv$ obey the product rule almost everywhere. In order that $uv$ has weak partial derivatives it is therefore sufficient to assume that the classical partial derivatives of $uv$ are locally integrable.”

5. **Proof.**

6. **Proof.**

7. **Proof.**
8. Proof.
11. Proof.
13. Proof.
15. Proof.
17. Proof.
18. Proof.