Elliptic PDEs of 2nd Order, Gilbarg and Trudinger

Chapter 6 Classical Solutions; the Schauder Approach

Yung-Hsiang Huang

1. Proof. (a) follows from (b), we only present a proof for (b) by mathematical induction here. We assume this is true for $k$ and try to prove it’s true for $k + 1$, that is, we assume

$$|a^{ij}_{k+1,\alpha;\Omega}|, |b^i_{k+1,\alpha;\Omega}|, |c^{(2)}_{k+1,\alpha;\Omega}| \leq \Lambda,$$

and there is some $C_k(n,\alpha,\lambda,\Lambda)$ such that for any open $V \subseteq \Omega$, if $Lv = f$ in $V$ with

$$|a^{ij}_{k,\alpha;V}|, |b^i_{k,\alpha;V}|, |c^{(2)}_{k,\alpha;V}| \leq \Lambda,$$

then

$$|v|^{(0)}_{k+2,\alpha;V} \leq C_k(n,\alpha,\lambda,\Lambda)(|u|_0 + |f|^{(2)}_{k,\alpha;V}). \quad (1)$$

Now we try to show

$$|u|^{(0)}_{k+3,\alpha;\Omega} \leq C_{k+1}(n,\alpha,\lambda,\Lambda)(|u|_0 + |f|^{(2)}_{k+1,\alpha;\Omega}).$$

for some constant $C_{k+1}(n,\alpha,\lambda,\Lambda)$.

Given $x \in \Omega$ and $B = B_{\frac{d_x}{2}}(x)$, we note that, for each $z \in B$,

(A) $\frac{d_z}{2} \leq d_z$, since $d_x \leq d(x,w) \leq d(x,z) + d(z,w) \leq \frac{d_x}{2} + d(z,w)$ for any $w \in \partial \Omega$.

(B) $d_{x,B} \leq \frac{d_x}{2} \leq d_x$, by (A).

To prove the desired inequality, we need to apply the interior Schauder estimate in the ball.

First, we note that by (B), $|p|^{(s)}_{m,B} \leq |p|^{(s)}_{m,\Omega}$ and by MVT and (A), $|p|^{(s)}_{m,\alpha;B} \leq |p|^{(s)}_{m+1,\alpha;\Omega}$ for each $0 \leq m \leq k, 0 \leq s$. Hence, we apply this result for $p = a^{ij}, b^i, c$ and $s = 0, 1, 2$ in the following without mentions.

Second, after differentiating the equation, we have for each $l = 1, 2, \cdots n$,

$$L(D_l u)(x) = -D_l a^{ij}(x) D_{ij} u(x) - D_l b^i(x) D_i u(x) - D_l c(x) u(x) + D_l f(x); \quad (2)$$

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†Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw
by the inductive assumption, the $k$-th interior Schauder’s estimates (1) to (2) in the ball $B$ implies that,

$$d_{x,B}|Du(x)| + d_{x,B}^2|D^2u(x)| + \cdots + d_{x,B}^{k+3}|D^{k+3}u(x)| \leq C_k(d_{x,B} \sup_{z \in B}|Du(z)| + d_{x,B}|D^{i_1}u(z)| + D\beta(z)D_i u(z) + Dc(z) u(z) - Df(z)|^{(2)}_{k,\alpha;B})$$

By (A), we know $d_{x,B} \sup_{z \in B}|Du(z)| \leq \sup_{z \in B}|Du(z)|d_z$ and by (A)(B)(1), for each $0 \leq m \leq k$,

$$d_{x,B}|D^{i_1}u(z)| + D\beta(z)D_i u(z) + Dc(z) u(z) - Df(z)|^{(2)}_{m,B} \leq \sup_{|\beta|=m} \sup_{z \in B} \sum_{\gamma \leq \beta} |D^\gamma D^{i_1}(z)|d_z^{\gamma+1}|D^{\beta-\gamma}D_i u(z)|d_z^{\beta-\gamma} + |D^\gamma D\beta(z)|d_z^{\gamma+2}|D^{\beta-\gamma}D_i u(z)|d_z^{\beta-\gamma+1}$$

$$+ |D^\gamma Dc(z)|d_z^{\gamma+3}|D^{\beta-\gamma}u(z)|d_z^{\beta-\gamma} + |D^{\beta}Df(z)|d_z^{m+3} \leq \sup_{|\beta|=m} \sum_{\gamma \leq \beta} (|a^{ij}|_{|\gamma|+1} + |b^{i}|_{|\gamma|+1} + |c^{j}|_{|\gamma|+1})(|u|^{(0)}_{|\beta-\gamma|+2} + |u|^{(0)}_{|\beta-\gamma|+1} + |u|^{(0)}_{|\beta-\gamma|} + |u|^{(0)}_{|\beta-\gamma|} + |f|^{(2)}_{m+1})$$

$$\leq \sup_{|\beta|=m} \sum_{\gamma \leq \beta} (|u|^{(0)}_{|\gamma|+1} + |b^{i}|_{|\gamma|+1} + |c^{j}|_{|\gamma|+1})(|u|^{(0)}_{|\beta-\gamma|+2} + |u|^{(0)}_{|\beta-\gamma|+1} + |u|^{(0)}_{|\beta-\gamma|} + |u|^{(0)}_{|\beta-\gamma|} + |f|^{(2)}_{m+1}) \leq [f]^{(2)}_{m+1} + \left(|a^{ij}|^{(0)}_{m+1} + |b^{i}|^{(1)}_{m+1} + |c^{j}|^{(2)}_{m+1}\right) \cdot 3|u|^{(0)}_{m+2} \leq [f]^{(2)}_{m+1} + 9\Lambda|u|^{(0)}_{m+2},$$
and

\[ d_{x,B}[Da^{ij}D_{ij}u(y) + Db^{i}D_{i}u(y) + Dcu(y) - Df(y)]_{k,\alpha:B}^{(2)} \]

\[ \leq \sup_{|\beta| = k} \sup_{y,z \in B, y \neq z} \left( \sum_{\gamma \leq \beta} \frac{|D^{\beta-\gamma} Da^{ij}(y)D^\gamma D_{ij}u(y) - D^{\beta-\gamma} Da^{ij}(z)D^\gamma D_{ij}u(z)|}{|y - z|^\alpha} \right) 
+ \frac{|D^{\beta-\gamma} Db^{i}(y)D^\gamma D_{i}u(y) - D^{\beta-\gamma} Db^{i}(z)D^\gamma D_{i}u(z)|}{|y - z|^\alpha} 
+ \frac{|D^{\beta-\gamma} Dc(y)D^\gamma u(y) - D^{\beta-\gamma} Dc(z)D^\gamma u(z)|}{|y - z|^\alpha} \] 

\[ \leq \sup_{|\beta| = k} \sup_{y,z \in \Omega, y \neq z} \left( \sum_{\gamma \leq \beta} \frac{|D^{\beta-\gamma} Da^{ij}(y) - D^{\beta-\gamma} Da^{ij}(z)|}{|y - z|^\alpha} d_{y,z:B}^{\gamma+1} |D^{\beta} D_{ij}u(y)| d_{y,z:B}^\gamma + 1 \right) 
+ \frac{|D^{\beta-\gamma} Db^{i}(y) - D^{\beta-\gamma} Db^{i}(z)|}{|y - z|^\alpha} d_{y,z:B}^{\gamma+1} |D^{\beta} D_{i}u(y)| d_{y,z:B}^\gamma + 1 \] 

\[ \leq \sup_{|\beta| = k} \left( \sum_{\gamma < \beta} [a^{ij}]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+2}^{(0)} + [a^{ij}]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+2}^{(0)} + [b^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} + [b^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} \right) 
+ \left( [c^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} + [c^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} \right) + \left( [f]_{k+1}^{(2)} \right) \] 

\[ \leq 2 \left( [a^{ij}]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+2}^{(0)} + [a^{ij}]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+2}^{(0)} + [b^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} + [b^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} \right) 
+ \left( [c^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} + [c^i]_{|\beta-\gamma|+1,\alpha:B} [u]_{|\gamma|+1}^{(0)} \right) 
+ \left( [f]_{k+1}^{(2)} \right) \] 

\[ \leq 2 \left( [a^{ij}]_{k+1,\alpha:B} + [b^i]_{k+1,\alpha:B} + [c^i]_{k+1,\alpha:B} + [f]_{k+1,\alpha:B} \right) \] 

Then

\[ |u|_{k+3,\alpha:B}^{(0)} \leq 2^{k+3} \left\{ |u|_{0} + C_k(C_k + (9 + 6)\Lambda C_k + 1)(|u|_{0} + |f|_{k+1,\alpha:B}^{(2)}) \right\} \leq C_{k+1}(|u|_{0} + |f|_{k+1,\alpha:B}^{(2)}) 
\]

where \( C_{k+1} := 2^{k+3}(1 + C_k + C_k^2 + 15\Lambda C_k^2) \).
3. One of the counterparts of exterior cone condition for parabolic equations is the exterior tusk condition. See Lieberman [1, Exercise 3.11] and Lorenz [2, Section 3.11.4].

\[ \text{Proof.} \]

4. If one go through the details of the construction of Perron solution, we will find out that the condition that \( \frac{b_i}{X} \) is bounded in \( \Omega \) is only used to show \( v^+ = \pm \sup_{\partial \Omega} |\varphi| \pm (e^{\gamma_d} - e^{\gamma_1}) \sup_{\Omega} \frac{|f|}{X} \) are super-(sub-)function of the Dirichlet problem \( Lu = f \) in \( \Omega, u = \varphi \) on \( \partial \Omega \). However, in this problem we know \( w^+ \equiv \pm \sup_{\partial \Omega} |\varphi| \) will be a super-(sub-) function even if \( \frac{b_i}{X} \) is unbounded.

\[ \text{Proof.} \] As mentioned above, the existence of Perron solution \( u(x) \) is examined in Section 6.3 and 6.6. To see \( u(x) \to \varphi(x_0) \) as \( x \to x_0 \), we follow the Remarks after Lemma 6.12 to establish \( w^+_\epsilon = \varphi(x_0) \pm \epsilon \pm k_i \nu(x_0) \cdot (x - x_0) \) as a local barrier relative to \( L, \varphi \) and \( \sup_{\Omega} |\varphi| \) at \( x_0 \) for some suitable positive constants \( k_i \). First we let \( B(x_0) =: B \) be the ball such that \( b \cdot \nu(x_0) \geq 0 \) in \( B \cap \Omega \). Then \( w^+_\epsilon(x_0) \to \varphi(x_0) \) as \( \epsilon \to 0 \) and \( w^+_\epsilon \) is a sub-(sup-)solution in \( \Omega \cap B \) since \( Lw^+_\epsilon = \pm k_i b(x) \cdot \nu(x_0) \).

Next, we check \( w^+_\epsilon \geq \sup_{\Omega} |\varphi| \) on \( \partial B \cap \Omega \) and \( w^+_\epsilon \geq \varphi \) on \( B \cap \partial \Omega \). (A sign changed argument for \( w^- \) part is omitted.) By uniform continuity of \( \varphi \), we know \( |\varphi(x) - \varphi(x_0)| < \epsilon \) if \( |x - x_0| \) is less than some \( \delta = \delta(\epsilon) \). By the strictly convexity of \( \Omega \) at \( x_0 \), we know \( \nu(x_0) \cdot (x - x_0) > 0 \) for all \( x \in \partial(\Omega \cap B) \setminus \{x_0\} \), and hence by the continuity, \( \nu(x_0) \cdot (x - x_0) \geq \tau > 0 \) on \( \partial(\Omega \cap B) \setminus B_\delta(x_0) \) for some \( \tau = \tau(\epsilon) \).

Now, we see that if we pick \( k_i = \frac{2 \sup_{\partial \Omega} |\varphi|}{\tau} \), then \( w^+_\epsilon = \varphi(x_0) + \epsilon + k_i \nu(x_0) \cdot (x - x_0) \geq \varphi(x) \) if \( x \in B_\delta(x_0) \cap \partial \Omega \) and \( \varphi(x_0) + \epsilon + 2 \sup_{\partial \Omega} |\varphi| \nu(x_0) \cdot (x - x_0) \geq - \sup |\varphi| + \epsilon + 2 \sup |\varphi| > \sup |\varphi| \) if \( x \in \partial(\Omega \cap B) \setminus B_\delta(x_0) \).

\[ \text{Remark} \] 1. Note that we only use \( b \cdot \nu(x_0) \geq 0 \) in showing \( w^+_\epsilon \) is sub-(super-)solution. Is it necessary to assume \( b \cdot \nu(x_0) > 0 \) in a neighborhood of \( x_0 \)?

5. We follows Michael [3].

\[ \text{Proof.} \]

6. We follows Michael [3, Section 5].

\[ \text{Proof.} \]

7. \[ \text{Proof.} \]
8. Proof.
10. This is based on Olejnik and Radkevic[4].

Proof.

11. Proof.

References


