Elliptic PDEs of 2nd Order, Gilbarg and Trudinger
Chapter 4  Poisson’s Equation and the Newtonian Potential

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1. Proof. (a) WLOG, we assume $\gamma = \min(\alpha, \beta) = \alpha$.

(1) Given $x, y \in \Omega$, then

$$|u(x)v(x) - u(y)v(y)| \leq |u(x)||v(x) - v(y)| + |v(y)||u(x) - u(y)| \leq \|v\|_0\|u\|_\beta|x-y|^\beta + \|v\|_0\|u\|_\alpha|x-y|^\alpha \leq |x-y|^\gamma \max(1, d^{\alpha+\beta-2\gamma}) \left( \|v\|_0\|u\|_\beta + \|v\|_0\|u\|_\alpha \right).$$

Hence, $[uv]_\gamma \leq \max(1, d^{\alpha+\beta-2\gamma}) \left( \|v\|_0\|u\|_\beta + \|v\|_0\|u\|_\alpha \right)$. Furthermore,

$$\|uv\|_\gamma \leq \max(1, d^{\alpha+\beta-2\gamma}) \left( \|v\|_0\|v\|_0 + \|v\|_0\|v\|_0 + \|v\|_0\|v\|_0 \right) \leq \max(1, d^{\alpha+\beta-2\gamma}) \|u\|_\alpha\|v\|_\beta.$$

(2) Given $x, y \in \Omega$, then

$$\frac{|u(x)v(x) - u(y)v(y)|}{|x-y|^\gamma} \leq \frac{|u(x)||v(x) - v(y)| + |v(y)||u(x) - u(y)|}{|x-y|^\gamma} \leq \|v\|_0\|u\|_\beta\frac{|x-y|^\beta}{|x-y|^\gamma} + \|v\|_0\|u\|_\alpha\frac{|x-y|^\alpha}{|x-y|^\gamma} = \|v\|_0\|u\|_\beta\frac{|v\|_\beta}{d^{\beta-\gamma}d^\gamma} + \|v\|_0\|u\|_\alpha\frac{|u\|_\alpha}{d^{\alpha-\gamma}d^\gamma} \leq \|v\|_0\|u\|_\beta\frac{|v\|_\beta}{d^\gamma} + \|v\|_0\|u\|_\alpha\frac{|u\|_\alpha}{d^\gamma}.$$ 

Hence, $[uv]_\gamma \leq \|v\|_0\|v\|_0 + \|v\|_0\|v\|_0$ and therefore $\|uv\|_\gamma \leq \|u\|_\alpha\|v\|_\beta$.

(b) Given $W \subset \subset \Omega$ and two distinct points $x, y \in W$, denote $I$ be the interval between $g(x)$ and $g(y)$ and $L_{f,I}$ be the Holder constant of $f$ on $I$, then

$$|f(g(x)) - f(g(y))| \leq L_{f,I}|g(x) - g(y)|^\alpha \leq L_{f,I}(L_{g,W}|x-y|^{\beta})^\alpha.$$ 

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2. Go back to check the convergence of the integrals in the proof of Lemma 4.2.

3. I think we need to assume $p > n$ rather than $p > n/2$. See Exercise 8. Also see Lieb and Loss, [3, Chapter 10].

**Proof.**

4. **Proof.**

5. **Proof.** Denote fundamental solution for Laplaian with pole $y$ by $\Gamma_y(x) := \Gamma(x - y)$. Put $v(x) = \Gamma_y(x) - \Gamma(R')$, where $0 < R' \leq R$, in the Green’s identity

$$
\int_{\Omega} u \Delta v - v \Delta u \, dx = \int_{\partial \Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, dS_x,
$$

with $\Omega = B_R(y) \setminus B_\epsilon(y)$ and letting $\epsilon \to 0$, we have

$$
u(y) = \int_{B_{R'}} \Delta u(y) [\Gamma_y(x) - \Gamma(R')] \, dx + \int_{\partial B_{R'}} u \frac{\partial \Gamma_y(x)}{\partial n} - [\Gamma_y(x) - \Gamma(R')] \frac{\partial u}{\partial n} \, dS_x
$$

$$= \left(\leq, \geq\right) \frac{1}{n w_n (R')^{n-1}} \int_{\partial B_{R'}} u \, dS_x + \int_{B_{R'}} f(x) [\Gamma_y(x) - \Gamma(R')] \, dx,$$

where $\Delta u = (\geq, \leq) f$. Next, we only consider the equality case since the other cases are similar.

Multiply both sides with $(R')^{-n}$ and integrate w.r.t $R'$ from 0 to $R$, we have, for $n > 3$,

$$u(y) = \frac{1}{w_n R^n} \int_{B_R(y)} u \, dx + \frac{1}{(2 - n) w_n R^n} \int_0^R \int_{B_{R'}(y)} f(x) \left[|x - y|^{2-n} - (R')^{2-n}\right] \, dx (R')^{-n} \, dR'
$$

$$= \frac{1}{w_n R^n} \int_{B_R(y)} u \, dx + \frac{1}{(2 - n) w_n R^n} \int_0^R \int_{\partial B_1(0)} f(y + r w) \, dr \, dR'
$$

$$= \frac{1}{w_n R^n} \int_{B_R(y)} u \, dx + \frac{1}{(2 - n) w_n R^n} \int_0^R \int_{\partial B_1(0)} \left[r^{2-n} (R')^{n-1} - R'\right] \, dr \, dR' f(y + r w) \, dw \, n^{-1} \, dr
$$

$$= \frac{1}{w_n R^n} \int_{B_R(y)} u \, dx + \frac{1}{n(2 - n) w_n} \int_0^R \int_{\partial B_1(0)} \left[r^{2-n} (R' - r^n) - \frac{1}{2} (R^2 - r^2)\right] \, dr \, dR'
$$

$$= \frac{1}{w_n R^n} \int_{B_R(y)} u \, dx + \frac{1}{2 - n} \int_{\partial B_1(0)} \left[Z^{2-n} + \frac{n - 2 R^2}{2 R^n} - \frac{n - 2 R^2}{2 R^n} - R^{2-n}\right] f(y + Z) \, dZ
$$

The calculation for $n = 2$ is similar.

6. **Proof.** (incomplete!!!!!) Since $\Omega$ is $C^2$ and bounded, we can find a neighborhood $\Gamma$ of $\partial \Omega$ in $\Omega$ such that $\text{dist}(x, \partial \Omega) =: d(x) \in C^2(\Gamma)$ and hence $\nabla d, \Delta d$ are bounded in $\Gamma$. Moreover, since $\Omega$ is compact, there exists $\delta > 0$ such that $B(x, \delta) \cap \Omega \subset \Gamma$ for every $x \in \Omega$. 

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Let $\eta \in C_0^\infty(B_\delta(0))$ with $0 \leq \eta \leq \eta(0) = (\beta(1-\beta))^{-1}$. Given $x \in \partial \Omega$ and let $\eta_x(y) := \eta(y-x)$.

Since
\[
\Delta(d^3\eta_x) = d^3\Delta\eta_x + 2\nabla d^3\nabla\eta_x + \Delta(d^3)\eta_x
\]
\[
= -d^3[-\beta(1-\beta)\eta_x - \beta d(\Delta d)\eta_x - 2\beta d \nabla d \nabla \eta_x - d^2 \Delta \eta_x],
\]
we can find a small $r > 0$ independent of $x$ such that for $y \in B_r(x)$, $\Delta(d^3\eta_x)(y) \leq \frac{1}{2}d^3$. On the other hand, for $y \notin B_r(x)$, $|\Delta(d^3\eta_x)(y)| \leq C(\beta, ||d||_{C^2(\Omega)}, r) =: C$ (Note $\eta_x = 0$ on $\partial \Omega \setminus B_{\delta}(x)$.)

Since $\partial \Omega$ is compact, there exists finite many $x_1, \ldots, x_m$ such that $\{B_r(x_i)\}_i$ covers $\partial \Omega$. Let $v$ be the solution of $\Delta v = -mC$ in $\Omega$ and $v = 0$ on $\partial \Omega$.

Define $w = \sum \eta_x d^3 + v$, then $w = 0$ on $\partial \Omega$ and $\Delta w \leq -\frac{1}{2}d^3$. So $\Delta(2Nw \pm u) \leq 0$ in $\Omega$ and $2Nw \pm u = 0$ on $\partial \Omega$. So $|u(x)| \leq 2Nw(x)$ in $\Omega$ by the maximum principle. It remains to estimate $v(x)$.

Note that since $|\nabla d(y)| \to 1$ as $y \to \partial \Omega$,
\[
\Delta(d^3)(y) = d(y)^{\beta-2}[\beta(\beta - 1)|\nabla d(y)|^2 + \beta d(y)\Delta d(y)] \to -\infty \text{ as } y \to \partial \Omega.
\]
So there exists a neighborhood $\Gamma' \subset \Gamma$ of $\partial \Omega$ and $C'$ such that
\[
\Delta(C'd^3 - v) \leq 0 \text{ in } \Gamma' \text{ and } C'd^3 - v \geq 0 \text{ on } \partial \Gamma'.
\]

By the maximum principle, $v(x) \leq C'd^3$ in $\Gamma'$.

7. Standard change of variables. I think this is the same as the derivation of Laplace-Beltrami operator in Riemannian geometry.

8. See Lieb and Loss, [3, Chapter 10].

Proof.

9. Proof. (a) Since $\Delta(\eta P) = (\Delta \eta)P + 2\nabla \eta \nabla P$, supp$(\Delta(\eta P)) \subset \{1 \leq |x| \leq 2\}$. Then for any $x \neq 0$ and $y \in B_{\frac{1}{2}|x|}(x)$, for all but finitely many $k$, $\Delta(\eta P)(t_k y) \neq 0$. So $f$ is continuous at any $x \neq 0$.

At the origin, we know $f(0) = 0$ from the definition. Since $|f(x)| = |c_k \Delta(\eta P)(t_k x)| \leq M|c_k|$ if $2^{-k} \leq x \leq 2^{-k+1}$ and $c_k \to 0$, $f$ is continuous at the origin.

Next, we define $v(x) = \sum c_k \eta P(t_k x)$. For each $x \neq 0$ and $y \in B_{\frac{1}{2}|x|}(x)$, we see only finite terms contribute $v(y)$ and hence $v \in C^2(\mathbb{R}^n \setminus \{0\})$. Since $\sum \frac{|c_k|}{t_k}$ converges and $\eta P$ is bounded, $v$ is continuous everywhere (and hence bounded near the origin).
Since for each \( x \neq 0 \), there is only one \( k_0 \) such that \( 1 \leq |2^{k_0}x| \leq 2 \), then for some \(|\alpha| = 2, D^\alpha P \equiv P_\alpha \neq 0 \) and
\[
\partial^\alpha v(x) = \sum_{k=0}^{k_0-1} c_k P_\alpha + c_{k_0} \eta(2^{k_0}x)P_\alpha + \sum_{i,\alpha_i=1} c_{k_0}(\partial^i \eta)(2^{k_0}x)(\partial^{\alpha_i}P)(2^{k_0}x)
\]

Since \( k_0(x) \to \infty \) as \(|x| \to 0\), \( c_{k_0}(x) \to 0 \) as \(|x| \to 0\). Moreover, since \( \sum c_k \) diverges, \( \lim_{|x| \to 0} \partial^\alpha v(x) \) does not exists.

Given \( \epsilon > 0 \). Suppose there exist classical solution to \( \Delta u = f \) in \( B_\epsilon \), then \( u - v \) is bounded harmonic in \( B_{\epsilon/2} \setminus \{0\} \). By removable singularity, we know \( u - v \) has a harmonic extension to the origin, which implies the contradiction that \( v \) has a \( C^2 \) extension to the origin.

(b) Similarly, we see \( w(x) := \sum_{k=0} \eta(Q)(t_k x) \) is \( C^3(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n) \). We also note that for each \( x \neq 0 \), \( \Delta w(x) = g(x) = \sum_{k=0} \eta(Q)(t_k x) \) and \( D_i g(x) = \sum_{k=0} (D_i \eta(Q))(t_k x) \), so \( g \in C^1(\mathbb{R}^n \setminus \{0\}) \). At the origin, we know \( g(0) = 0 \) from the definition and for each \( h \neq 0 \), there is only one \( k_0 \) such that \( 1 \leq |2^{k_0}h| \leq 2 \). Note that \( k_0(h) \to \infty \) as \( h \to 0 \) and
\[
|\frac{g(he_i) - g(0)}{h}| = |c_{k_0} \eta(Q)(2^{k_0}h e_i)| \leq |c_{k_0}| M \to 0 \text{ as } h \to 0.
\]

So \( D_i g(0) = 0 \). Since \( |D_i g(x)| = |c_{k_i} D_i \eta(Q)(t_k x)| \leq M' |c_i| \) if \( x \in [2^{-k}, 2^{-k+1}] \) and \( c_k \to 0 \), \( D_i g \) is continuous at the origin for each \( i \). Therefore, \( g \in C^1(\mathbb{R}^n) \).

Since for each \( x \neq 0 \), there is only one \( k_0 \) such that \( 1 \leq |2^{k_0}x| \leq 2 \), then for some \(|\alpha| = 3, D^\alpha Q \equiv Q_\alpha \neq 0 \) and
\[
\partial^\alpha w(x) = \sum_{k=0}^{k_0-1} c_k Q_\alpha + c_{k_0} \eta(2^{k_0}x)Q_\alpha + \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta! (\alpha - \beta)!} (\partial^\beta \eta)(2^{k_0}x)(\partial^{\alpha - \beta} P)(2^{k_0}x)
\]

Since \( k_0(x) \to \infty \) as \(|x| \to 0\), \( c_{k_0}(x) \to 0 \) as \(|x| \to 0\). Moreover, since I assume \(|\sum c_k| = \infty\), \( \lim_{|x| \to 0} |\partial^\alpha w(x)| = \infty \) and hence \( w \) is not \( C^{2,1} \) in any neighborhood of the origin by the mean value theorem (MVT).

\[\square\]

**Remark** 1. Another example is given in [2] Section 3.4 where \( u = (x_1^2 - x_2^2)(-\log |x|)^{1/2} \) on \( B_R(0), R < 1 \).

**Remark** 2. This problem is concern the existence of \( C^2(\Omega) \) solution to Dirichlet problem in \( B_1 \). Another problem one may ask is whether the \( C^2 \)-global regularity theorem true? That is, if \( u \in C^2(B_1) \cap C^0(\overline{B_1}) \) solves \( \Delta u = f \in C^0(\overline{B_1}) \) and \( u = g \in C^2(\overline{B_1}) \), can we conclude that \( u \in C^2(\overline{B_1}) \) ?

This question is related to the analytic continuation, I find it’s answered negatively in [1] Chapter II.3. The example is the following:
Consider a conformal map $f : D \subset \mathbb{C} \to \Omega$ where $\Omega = \{x + iy : 0 < x < \frac{1}{1+|y|}\}$. Clearly, $f$ is unbounded. On the other hand, Re $f$ has a continuous extension to $\overline{D}$ because it has a finite limit. Write $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and define $F(z) = \sum_{n=1}^{\infty} c_n n^{-2} z^n$. Re$F$ will be the counterexample. The reason is:

If all the second partial derivatives of Re$F$ are bounded on $D$, then $F''$ is bounded by Cauchy-Riemann equations. But this is impossible since $f(z) - f(0) = z (z F')' = z F' + z^2 F''$ where the left hand side is unbounded and the right hand side is bounded by MVT.

10. I think the denominator in (a) should be $2(n - 2)$, not $2n$. (Of course, this is for $n \geq 3$, and for $n = 2$, we use the same technique as the proof for Theorem 4.6 to show the denominator can be $2(3 - 2) = 2$). For example, take radial function $f = f(r) \in C_c^\infty(B_R(0))$ such that $-1 \leq f \leq 0$, $f \equiv -1$ on $r \leq R - 2\epsilon$ and $f \equiv 0$ on $r > R - \epsilon$. Then $|u(0)| = \int_B \frac{|x - y|^{2-n} f(y)}{nw_n(2-n)} dy = \int_0^R r f(r) dr / (2 - n) \in \left( \frac{R^2}{2(n-2)}, \frac{(R-\epsilon)^2}{2(n-2)} \right)$.

Proof. Since $u \in C_0^2(B), f \in C_0(B)$ and hence for $n \geq 3$ and for each $x \in B$,

$$|u(x)| = \left| \int_B \frac{|x - y|^{2-n} f(y)}{nw_n(2-n)} dy \right| \leq |f|_0 \frac{R^2}{2(n-2)}$$

On the other hand, for each $x \in B$ and for $n \geq 2$,

$$|D_i u(x)| = \left| \int_B \frac{|x - y|^{-n}(x_i - y_i) f(y)}{nw_n} dy \right| \leq \int_{B_R(x)} \frac{|x - y|^{1-n}}{nw_n} dy |f|_0 = R |f|_0.$$

References

