4.1 Topological Spaces

1. If $\text{card}(X) \geq 2$, there is a topology on $X$ that is $T_0$ but not $T_1$.

   \textit{Proof.} \hfill \Box

2. If $X$ is an infinite set, the cofinite topology on $X$ is $T_1$ but not $T_2$, and is first countable iff $X$ is countable.

   \textit{Proof.} \hfill \Box

3. Every metric space is normal.

   \textit{Proof.} \hfill \Box

4. Let $X = \mathbb{R}$, and let $\mathcal{T}$ be the family of all subsets of $\mathbb{R}$ of the form $U \cup (V \cap \mathbb{Q})$ where $U, V$ are open in the usual sense. Then $\mathcal{T}$ is a topology that is Hausdorff but not regular. (In view of Exercise 3, this shows that a topology stronger than a normal topology need not be normal or even regular.)

   \textit{Proof.} \hfill \Box

5. Every separable metric space is second countable.

   \textit{Proof.} \hfill \Box

---

*Last Modified: 2017/10/13.

†Department of Math., National Taiwan University. Email: d04221001@ntu.edu.tw
6. Let $\mathcal{E} = \{(a, b) : -\infty < a < b < \infty\}$.

(a) $\mathcal{E}$ is a base for a topology $\mathcal{T}$ on $\mathbb{R}$ in which the members of $\mathcal{E}$ are both open and closed.

(b) $\mathcal{T}$ is first countable but not second countable. (If $x \in \mathbb{R}$, every neighborhood base at $x$ contains a set whose supremum is $x$.)

(c) $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathcal{T}$. (Thus the converse of Proposition 4.5 is false.)

Proof.

7. If $X$ is a topological space, a point $x \in X$ is called a cluster point of the sequence $\{x_j\}$ if for every neighborhood $U$ of $x, x_j \in U$ for infinitely many $j$. If $X$ is first countable, $x$ is a cluster point of $\{x_j\}$ iff some subsequence of $\{x_j\}$ converges to $x$.

Proof.

8. If $X$ is an infinite set with the cofinite topology and $\{x_j\}$ is a sequence of distinct points in $X$, then $x_j \to x$ for every $x \in X$.

Proof.

9. If $X$ is a linearly order set, the topology $\mathcal{T}$ generated by the sets $\{x : x < a\}$ and $\{x : x > a\} (a \in X)$ is called the order topology.

(a) If $a, b \in X$ and $a < b$, there exist $U, V \in \mathcal{T}$ with $a \in U, b \in V$, and $x < y$ for all $x \in U$ and $y \in V$. the order topology is the weakest topology with this property.

(b) If $Y \subset X$, the order topology on $Y$ is never stronger than, but may be weaker than, the relative topology on $Y$ induced by the order topology on $X$.

(c) The order topology on $\mathbb{R}$ is the usual topology.

Proof.

10. A topological space $X$ is called disconnected if there exist nonempty open sets $U, V$ such that $U \cap V = \emptyset$ and $U \cup V = X$; otherwise, $X$ is connected. When we speak of connected or disconnected subsets of $X$, we refer to the relative topology on them.
(a) $X$ is connected iff $\emptyset$ and $X$ are the only subsets of $X$ that are both open and closed.

(b) If $\{E_a\}_{a \in A}$ is a collection of connected subsets of $X$ such that $\cap_{a \in A}E_a \neq \emptyset$, then $\cup_{a \in A}E_a$ is connected.

(c) If $A \subset X$ is connected, then $\overline{A}$ is connected.

(d) Every point $x \in X$ is contained in a unique maximal connected subset of $X$, and this subset is closed. (It is called the connected component of $x$)

Proof. (a) is trivial. (b) Let $E := \cup_{a \in A}E_a$. Given $O_1, O_2$ open in $X$ such that $O_1 \cap E, O_2 \cap E$ are non-empty, $O_1 \cup O_2 \supset E$. Choose $x \in \cap_{a \in A}E_a$, and WLOG assume $x \in O_1$. Choose $y \in O_2 \cap E$, then $y \in E_\beta$ for some $\beta \in A$. Note that $(O_1 \cap E_\beta) \cup (O_2 \cap E_\beta) = (O_1 \cup O_2) \cap E_\beta = E_\beta$ and $x \in O_1 \cap E_\beta, y \in O_2 \cap E_\beta$. By the connectedness of $E_\beta, O_1 \cap O_2 \neq \emptyset$. Since $O_1, O_2$ are arbitrary chosen, $E$ is connected.

(c) Given $O_1, O_2$ open in $X$ such that $O_1 \cap \overline{A}, O_2 \cap \overline{A}$ are non-empty, $O_1 \cup O_2 \supset \overline{A} \supset A$. Given $x \in O_1 \cap \overline{A}$, since $O_1$ is an open neighborhood of $x$, $O_1 \cap A \neq \emptyset$. Similar for $O_2 \cap A \neq \emptyset$. By the connectedness of $A, O_1 \cap O_2 \neq \emptyset$. Since $O_1, O_2$ are arbitrary chosen, $\overline{A}$ is connected.

(d) Let $\mathcal{C} := \{A \subset X : A$ is connected and contains $x\}$. Let $C = \cup_{A \in \mathcal{C}}A$. By (b), $C$ is connected. Then it’s obvious maximal, unique. The closedness follows from (c).

11. If $E_1, \cdots E_n$ are subsets of a topological space, then $\bigcup_{i=1}^n E_j = \bigcup_{i=1}^n \overline{E_j}$.

(It’s not true for intersection, e.g. $E_1 = \mathbb{Q}$ and $E_2 = \mathbb{Q}^c$; it’s also not true for infinitely many $E_j$, e.g. $E_j = \{\frac{1}{j}\}$.)

Proof. This is easy, we omit it.

12. Let $X$ be a set. A Kuratowski closure operator on $X$ is a map $A \mapsto A^*$ from $\mathcal{P}(X)$ to itself satisfying (i) $\emptyset^* = \emptyset$, (ii) $A \subset A^*$ for all $A$, (iii) $(A^*)^* = A^*$ for all $A$, and (iv) $(A \cup B)^* = A^* \cup B^*$ for all $A, B$.

(a) If $X$ is a topological space, the map $A \mapsto \overline{A}$ is a Kuratowski closure operator.

(b) Conversely, given a Kuratowski closure operator, let $\mathcal{F} = \{A \subset X : A = A^*\}$ and $\mathcal{I} = \{U \subset X : U^c \in \mathcal{F}\}$. Then $\mathcal{I}$ is a topology, and for any set $A \subset X, A^*$ is its closure with respect to $\mathcal{I}$.

Proof.
13. If $X$ is a topological space, $U$ is open in $X$ and $A$ is dense in $X$, then $\overline{U} = \overline{U \cap A}$. (It’s not true if $U$ is not open, e.g. $X = \mathbb{R}, U = \mathbb{R} \setminus \mathbb{Q}$ and $A = \mathbb{Q}$.)

Proof. Of course, $U \supset \overline{U \cap A}$.

Given $x \in U$, if $x \in A$, then $x \in U \cap A$. If $x \notin A$, then $x \in \text{acc}(A)$. Given $V$ open in $X$, since $U$ is open, $V \cap U$ is open in $X$ and hence there is a point $y$ lying in $A \cap (V \cap U) = V \cap (A \cap U)$ and hence $y \neq x$. Since $V$ is given, $x \in \text{acc}(A \cap U)$. Consequently, $x \in \overline{U \cap A}$.

Since $x$ is given, $U \subset \overline{U \cap A}$. Since $\overline{U \cap A}$ is closed, $\overline{U} \subset \overline{U \cap A}$. \hfill \Box

4.2 Continuous Maps

14. Proof. \hfill \Box
15. Proof. \hfill \Box
16. Proof. \hfill \Box
17. Proof. \hfill \Box
18. Proof. \hfill \Box
19. Proof. \hfill \Box
20. Proof. \hfill \Box
21. Proof. \hfill \Box
22. Proof. \hfill \Box
23. Proof. \hfill \Box
24. Proof. \hfill \Box
25. Proof. \hfill \Box
26. Proof. \hfill \Box
27. Proof. \hfill \Box
28. Let $X$ be a topological space equipped with an equivalence relation, $\tilde{X}$ the set of equivalence classes, $\pi : X \to \tilde{X}$ the map taking each $x \in X$ to its equivalence class, and $\mathcal{T} = \{ U \subset \tilde{X} : \pi^{-1}(U) \text{ is open in } X \}$.

(a) $\mathcal{T}$ is a topology on $\tilde{X}$. (It is called the quotient topology.)

(b) If $Y$ is a topological space, $f : \tilde{X} \to Y$ is continuous iff $f \circ \pi$ is continuous.

(c) $\tilde{X}$ is $T_1$ iff every equivalence class is closed in $X$.

Proof. (a) Since $\pi^{-1}(\emptyset_{\tilde{X}}) = \emptyset_X$ and $\pi^{-1}(\tilde{X}) = X$, $\emptyset_X$ and $\tilde{X}$ are in $\mathcal{T}$. Given $\{U_\alpha\} \subset \mathcal{T}$, then

$$\pi^{-1}\left( \bigcup_{\alpha} U_\alpha \right) = \bigcup_{\alpha} \pi^{-1} U_\alpha$$

hence $\bigcup_{\alpha} U_\alpha \in \mathcal{T}$. Similarly $\cap_1^n U_i \in \mathcal{T}$ for every finite set $\{U_1, \cdots U_n\} \subset \mathcal{T}$.

(b) By definition, $\pi$ is continuous. So $f \circ \pi$ is continuous provided $f$ is. Conversely, if $f \circ \pi$ is continuous, then for every open set $V \subset Y$, $f^{-1}(V)$ is open in $\tilde{X}$ since $\pi^{-1} f^{-1}(V) = (f \circ \pi)^{-1}(V)$ is open in $X$. Therefore, $f$ is continuous.

(c) Suppose $\tilde{X}$ is $T_1$. Given any equivalence class $\pi^{-1}([x])$ for some $x \in \tilde{X}$, since the singleton $\{x\}$ is closed, by continuity, $\pi^{-1}([x])$ is closed.

Conversely, suppose every equivalence class is closed. Given $[x] \in \tilde{X}$, then $\pi^{-1}([x])$ is closed. Since $\pi^{-1}([x])^c = \left( \pi^{-1}([x]) \right)^c$ is open, $[x]^c \in \mathcal{T}$. So $\{x\}$ is closed. Hence $\tilde{X}$ is $T_1$.

29. Proof.

4.3 Nets

30. Proof.

31. Proof.

32. Proof.

33. Proof.

34. Proof.

35. Proof.

36. Proof.
4.4 Compact Spaces

37. Proof. □
38. Proof. □
40. Proof. □
41. Proof. □
42. Proof. □
43. Proof. □
44. Proof. □
45. Proof. □

4.5 Locally Compact Hausdorff Spaces

46. Proof. □
47. Proof. □
48. Proof. □
49. Proof. □
50. Proof. □
51. Proof. □
52. Proof. □
53. Proof. □
54. Proof. □
55. Proof. □
56. Proof. □
57. Proof. □
4.6 Two Compactness Theorems

58. Proof.

59. Proof.

60. Proof.

61. Proof.


63. Proof.

64. By Arezla-Ascoli Theorem.

65. Proof.

4.7 The Stone-Weierstrass Theorem

66. Proof.

67. Proof.

68. Proof.

69. Proof.

70. Proof.

71. Proof.

References