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A novel analytical power series solution for solute transport in a radially convergent flow field

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Abstract

The concentration breakthrough curves at a pumping well for solute transport in a radially convergent flow field are governed by an advective–dispersive second order partial differential equation with a radial distance-dependent velocity and dispersion coefficient. The Laplace transform is generally first employed to eliminate the temporal derivative to solve the partial differential equation. The Laplace transformed equations are then converted to the standard form of the special Airy function through successive applications of variable change. This study presents the solution of the Laplace-transformed equation without using the special Airy function. A direct power series method and a power series method with variable changes to eliminate the advection term that usually results in numerical errors for large Peclet numbers are applied to obtain an analytical solution in the Laplace domain. The obtained solutions are compared to other Airy function-formed solutions to examine the method's robustness and accuracy. Analytical results indicate that the Laplace transform power series method with variable change can effectively and accurately handle the radial advection–dispersion equation of high Peclet numbers, whereas the direct power series method can only evaluate the solution for medium Peclet numbers. The novel power series technique with variable change is valuable for future quantitative hydrogeological issues with variable dependent differential equation and can be extended to higher dimensional problems.

Keywords: Solute transport; Power series; Analytical solution; Radially convergent tracer test

1. Introduction

The radial dispersion problem refers to the problem of analyzing the dispersive transport of a contaminant or tracer in steady radial flow from a recharge well or pumping well that fully penetrates an homogeneous confined aquifer of uniform thickness and infinite lateral extent. Beyond its obvious importance in the studying solute transport for tracer tests in a radially divergent/convergent flow field or in aquifer decontamination by pumping, the radial dispersion problem is distinguished by its being probably the simplest case for which the dispersion coefficient is a function of a spatially varying velocity field. Accordingly, the analytical solution to the problem can be a valuable means of checking the accuracy of computer codes that simulate solute transport in porous media. Deriving the analytical solution of the differential equation in cylindrical coordinates that describe the radial dispersion problem is very difficult due to the dependence of the hydrodynamic dispersion

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Table 1
Dimensionless parameters

Dimensionless quantity	Expression
Time[a]	$t_D = \frac{t}{t_a}$
Distance	$r_D = \frac{r}{r_L}$
Pumped well radius	$r_{wD} = \frac{r_w}{r_L}$
Peclet number	$Pe = \frac{r_L}{a_L}$
Concentration[b] (slug input)	$C_D = \frac{C}{C_i}$
Concentration (continuous input)	$C_D = \frac{C}{\varepsilon C_0}$
Pumped well mixing factor	$\mu_w = \frac{r_w^2 h_w}{\phi h(r_L^2 - r_w^2)}$
Injection well mixing factor	$\mu_i = \frac{r_i^2 h_i}{\phi h(r_L^2 - r_w^2)}$

[a] $t_a = \pi h\phi(r_L^2 - r_w^2)/q_0$.
[b] $C_i = M/[\pi h\phi(r_L^2 - r_w^2)]$.

coefficient on the spatially varying velocity. Several researchers have attempted to derive the analytical solution of the radial advection–dispersion differential equations. The Laplace transform is normally the first manipulation to be employed to eliminate the temporal derivative when solving the partial differential equations. Tang and Babu (1979) presented an exact analytical solution for the divergent radial dispersion problem. Their solution, however, involved numerical integrations that were difficult to evaluate. Moench and Ogata (1981) obtained the quasi-analytical solutions by numerically inverting the Laplace transform.

Chen (1985) determined the analytical solution of the radial dispersion problem, by showing how to transform the governing advection–dispersion equation to the standard form of the Airy equation through successive applications of variable change. Following Chen's work, several researchers have obtained similar closed-form analytical or semi-analytical solutions expressed as the special Airy function for both conservative and sorbing solute transport in the divergent/convergent tracer test, for waste water injection into an aquifer and for aquifer decontamination by pumping (Chen, 1987; Chen and Woodside, 1988; Moench, 1989, 1991, 1995; Goltz and Oxley, 1991; Harvey et al., 1994; Haggerty and Gorelick, 1995; Chen et al., 1996; Becker and Charbeneau, 2000).

The Laplace transformed analytical solution for radially advective–dispersive differential equation can be expressed as the product of an Airy function and an exponential function (Chen, 1985), both of which may be represented as infinite power series. The power series method is the standard approach for solving linear differential equations with variable coefficients. The power series method of solving differential equations is straightforward in principle (Wylie and Barrett, 1995; Kreyszig, 1998). To our knowledge, no researchers have applied power series techniques to solve the radial advection–dispersion differential equations. The aim of this work then is to establish a general Laplace transform power series technique to solve the radial advection–dispersion differential equation with a spatially dependent variable coefficient. The solution of the new analytical power series will be verified by Moench's (1989) solution. The mathematical behavior of the new solutions will also be analyzed and compared with that of the special Airy function. The novel power series method can be extended to more complex, and multi-dimensional hydrological problem with variable dependent coefficient differential equation.

2. Problem description

The one-dimensional, convergent radial dispersion transport, investigated by Moench (1989), is considered in evaluating the applicability and accuracy of the power series method for a radial advection–dispersion problem. Moench (1989) thoroughly specified the assumptions made in the mathematical statement of the problem. The role of governing equations, boundary conditions and analytical solution will be briefly described.

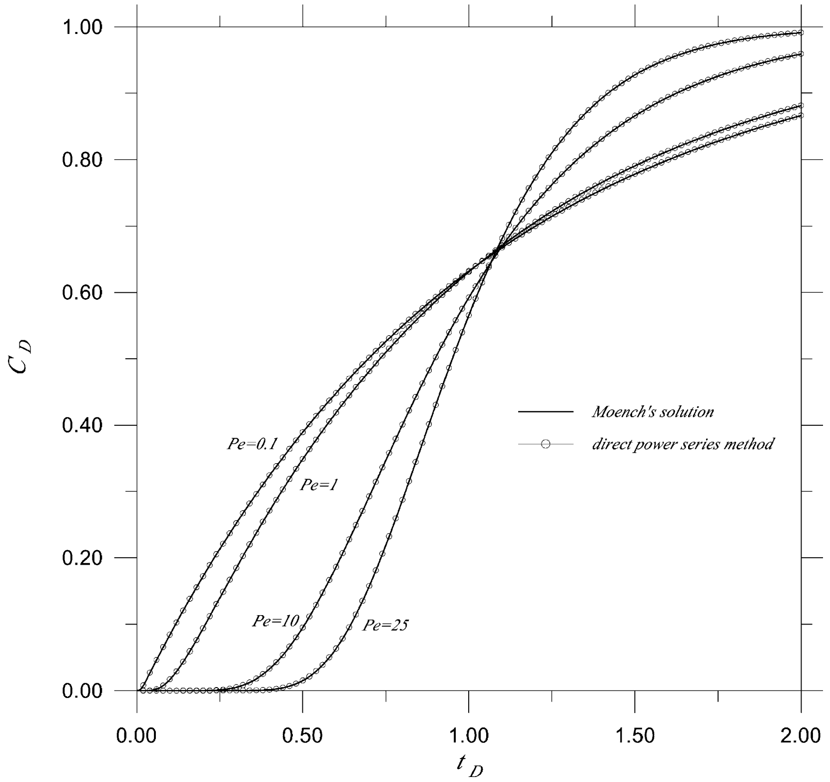

Fig. 1. Comparison of dimensionless breakthrough curves for a step tracer input without well bore mixing effect of the direct power seires solution and Moench's solution.

The dimensionless mathematical model for this case is

$$\frac{1}{Pe}\frac{\partial^2 C_D}{\partial r_D^2}+\frac{\partial C_D}{\partial r_D}=\frac{2r_D R}{(1-r_{wD}^2)}\frac{\partial C_D}{\partial t_D}, \tag{1}$$

$$r_{wD}\le r_D\le 1$$

$$C_D(r_D,0)=0 \tag{2}$$

$$\frac{1}{Pe}\frac{\partial C_D}{\partial r_D}=\mu_w\frac{\partial C_D}{\partial t_D}, \qquad r_D=r_{wD} \tag{3}$$

$$\frac{1}{Pe}\frac{\partial C_D}{\partial r_D}=\Gamma-\mu_i\frac{\partial C_D}{\partial t_D}, \qquad r_D=1 \tag{4}$$

where Pe is the Peclet number; C_D is the dimensionless concentration; t_D is the dimensionless time; r_D is the dimensionless radial distance; r_{wD} is the dimensionless radius of the pumped well; μ_w is pumped well mixing factor; μ_i is the injection well mixing factor; $\Gamma=\delta(\,)$ for a slug input; $\Gamma=1$ for a continuous input, and $\delta(\,)$ is the Dirac delta function. Table 1 defines the dimensionless variables.

Applying the Laplace transform with respect to t_D in Eqs. (1)–(4) yield

$$\frac{1}{Pe}\frac{d^2\bar{C}_D}{dr_D^2}+\frac{d\bar{C}_D}{dr_D}=\frac{2r_D R}{(1-r_{wD}^2)}s\bar{C}_D, \tag{5}$$

$$r_{wD}\le r_D\le 1$$

$$\frac{1}{Pe}\frac{\partial\bar{C}_D}{\partial r_D}=\mu_w s\bar{C}_D, \qquad r_D=r_{wD} \tag{6}$$

$$\frac{1}{Pe}\frac{\partial\bar{C}_D}{\partial r_D}=\psi-\mu_i s\bar{C}_D, \qquad r_D=1 \tag{7}$$

where $\bar{C}_D$ is the Laplace transform of C_D; s is the Laplace transform parameter; $\psi=1$ for a slug input, and $\psi=1/s$ for continuous input.

For this radial advection–dispersion problem, Moench (1989) used an Airy function to derive the analytical solution in the Laplace domain. The

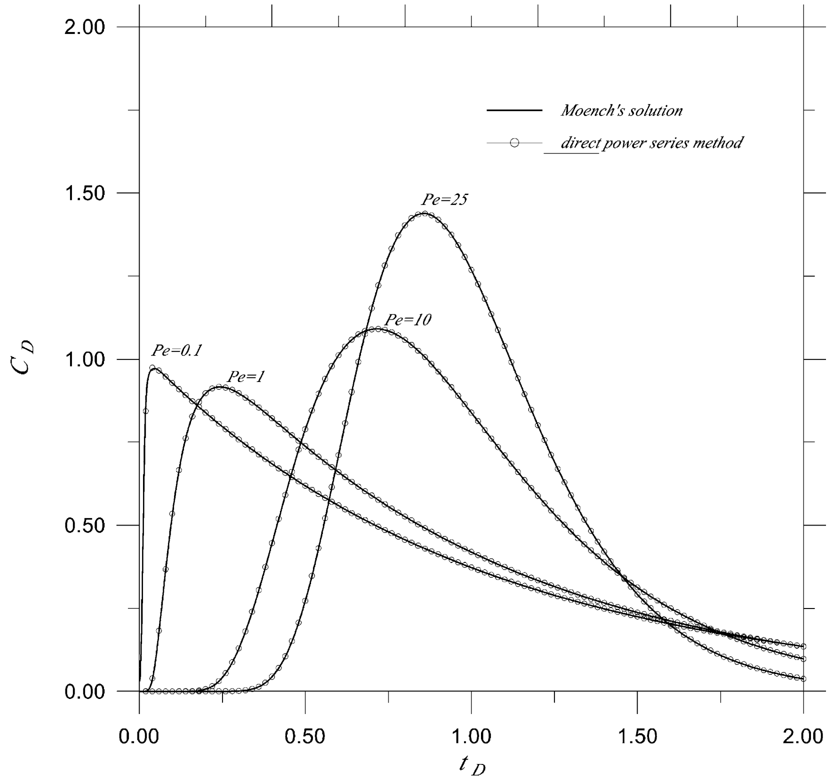

Fig. 2. Comparison of dimensionless breakthrough curves for a Dirac delta tracer input without well bore mixing effect of the power series solution and Moench's solution.

Laplace transform particular solution at the pumped well may be written as

$$\bar{C}_{\mathrm{D}}(r_{\mathrm{wD}}, s) = \psi \exp\left[\frac{Pe}{2}(1 - r_{\mathrm{wD}})\right]\bar{G}(s) \tag{8}$$

where

$$\bar{G}(s) = \frac{A_1B_2 - A_2B_1}{F + F_{\mathrm{w}} + F_{\mathrm{i}} + F_{\mathrm{iw}}};$$

$$F = \sigma^{1/3}(A_1B_3 - A_3B_1) + 0.5(A_3B_2 - A_2B_3)$$
$$+0.5(A_1 - B_2) + 0.25\sigma^{-1/3}(B_2 - A_2);$$

$$F_{\mathrm{w}} = \mu_{\mathrm{w}}s\left[(A_3B_2 - A_2B_3) + 0.5\sigma^{-1/3}(B_2 - A_2)\right];$$

$$F_{\mathrm{i}} = \mu_{\mathrm{i}}s\left[(A_1 - B_1) + 0.5\sigma^{-1/3}(B_2 - A_2)\right];$$

$$F_{\mathrm{iw}} = \mu_{\mathrm{i}}\mu_{\mathrm{w}}s^2\sigma^{-1/3}(B_2 - A_2);$$

$$A_1 = \left[\mathrm{Ai}\left(\sigma^{1/3}y_0\right)\right]'/\mathrm{Ai}\left(\sigma^{1/3}y_{\mathrm{L}}\right);$$

$$A_2 = \mathrm{Ai}\left(\sigma^{1/3}y_0\right)/\mathrm{Ai}\left(\sigma^{1/3}y_{\mathrm{L}}\right);$$

$$A_3 = \left[\mathrm{Ai}\left(\sigma^{1/3}y_{\mathrm{L}}\right)\right]'/\mathrm{Ai}\left(\sigma^{1/3}y_{\mathrm{L}}\right)$$

$$B_1 = \left[\mathrm{Bi}\left(\sigma^{1/3}y_0\right)\right]'/\mathrm{Bi}\left(\sigma^{1/3}y_{\mathrm{L}}\right);$$

$$B_2 = \mathrm{Bi}\left(\sigma^{1/3}y_0\right)/\mathrm{Bi}(\sigma^{1/3}y_{\mathrm{L}});$$

$$B_3 = \left[\mathrm{Bi}\left(\sigma^{1/3}y_{\mathrm{L}}\right)\right]'/\mathrm{Bi}\left(\sigma^{1/3}y_{\mathrm{L}}\right)$$

$$y_0 = Pe\ r_{\mathrm{wD}} + 0.25\sigma^{-1};\ \ y_{\mathrm{L}} = Pe + 0.25\sigma^{-1}$$

3. Power series solution

The transformed ordinary differential Eq. (5) can be directly solved by the power series method. The

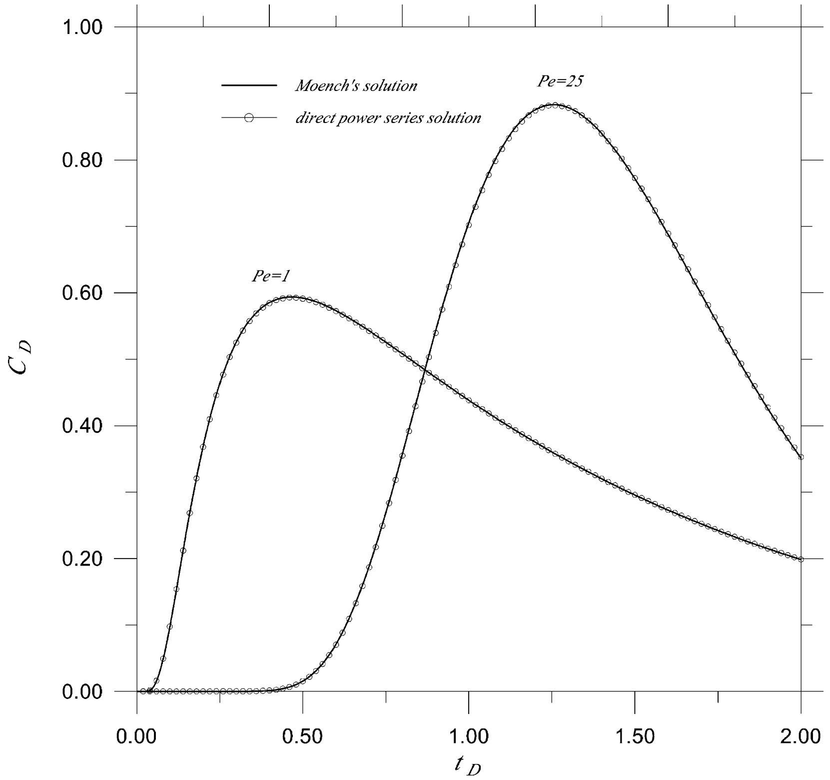

Fig. 3. Comparison of dimensionless breakthrough curves for a step tracer input with well bore mixing effect (μ_w, $\mu_i = 0.25$) of the direct power series solution and Moench's solution.

first spatial derivative of Eq. (5), however, is often generally a source of numerical error for large Peclet numbers (Xu and Brusseau, 1995). Accordingly, two power series solution approaches are presented and examined to evaluate the numerical error that results from large Peclet numbers. One solution is obtained by directly applying the power series method. The other solution is determined using the power series method after a variable change is employed to eliminate the first spatial derivative.

3.1. Direct power series solution

The solution of Eq. (5) is assumed to be in the form of a power series with unknown coefficients. The power series solution of governing Eq. (5) subject to boundary conditions (6) and (7), then can be derived as

$$\bar{C}_D = \sum_{m=0}^{\infty} a_m r_D^m \tag{9}$$

and substitute this series along with the series obtained by term-wise differentiation of Eq. (9),

$$\frac{\partial \bar{C}_D}{\partial r_D} = \sum_{m=1}^{\infty} m a_m r_D^{m-1} \tag{10}$$

$$\frac{\partial^2 \bar{C}_D}{\partial r_D^2} = \sum_{m=2}^{\infty} m(m-1) a_m r_D^{m-2} \tag{11}$$

into Eq. (5), yielding the following equation

$$\frac{1}{Pe}\sum_{m=2}^{\infty} m(m-1) a_m r_D^{m-2} + \sum_{m=1}^{\infty} m a_m r_D^{m-1} - \frac{2\mathrm{Rs}}{1-r_{wD}} r_D \sum_{m=0}^{\infty} a_m r_D^m = 0 \tag{12}$$

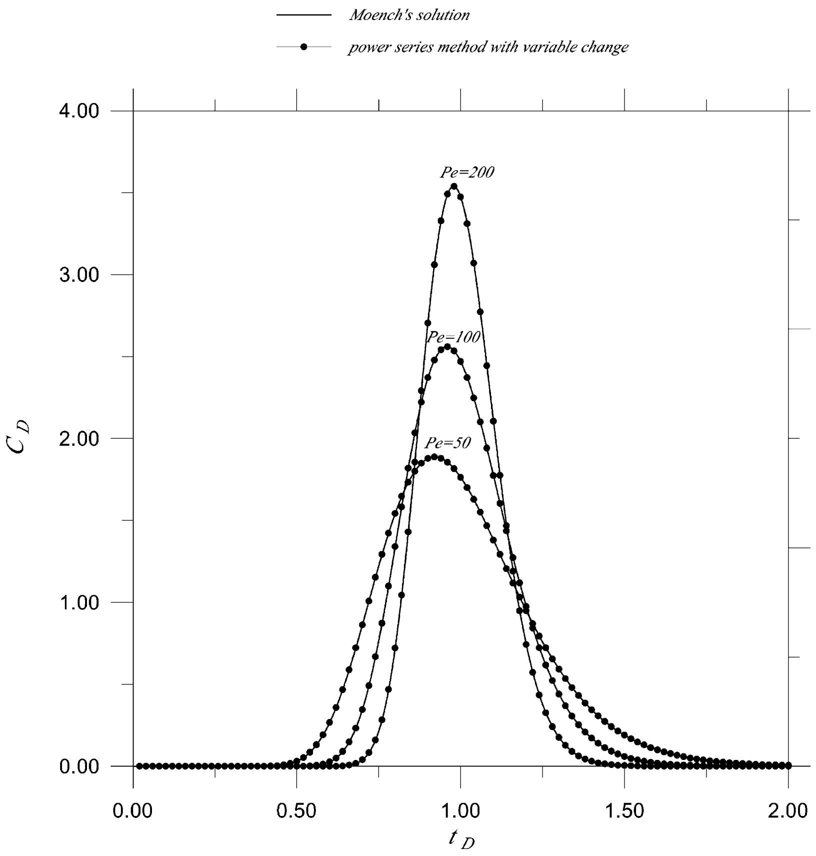

Fig. 4. Comparison of dimensionless breakthrough curves for a step tracer input without well bore mixing effect of the power series solution with variable change and Moench's solution for large Peclet numbers.

Shifting the summation indices of Eq. (12) yields

$$\frac{1}{Pe}\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}r_{\mathrm{D}}^{m}+\sum_{m=0}^{\infty}(m+1)a_{m+1}r_{\mathrm{D}}^{m} - \frac{2\mathrm{Rs}}{1-r_{\mathrm{wD}}^{2}}\sum_{m=1}^{\infty}a_{m-1}r_{\mathrm{D}}^{m} = 0 \tag{13}$$

Setting the coefficients of each power of r_{D} to zero, for $m=0$,

$$a_2 = \frac{1}{2A}a_0 \tag{14a}$$

and in general, when $m = 1,2,3,\ldots,$

$$a_{m+2} = -\frac{m+1}{Pe(m+2)(m+1)}a_{m+1} + \frac{2\mathrm{Rs}}{Pe\left(1-r_{\mathrm{wD}}^{2}\right)(m+2)(m+1)}a_{m-1} \tag{14b}$$

the general solution can be yielded by inserting the appropriate values of the coefficients into the following equation:

$$\bar{C}_{\mathrm{D}}(r_{\mathrm{D}},s) = b_1F_1(r_{\mathrm{D}},s) + b_2F_2(r_{\mathrm{D}},s) \tag{15}$$

where $F_1(r_{\mathrm{D}},s)$ and $F_2(r_{\mathrm{D}},s)$ are two linearly independent general functions in the form of infinite series as in Eq. (9). The values of the series coefficients (a_m) of $F_1(r_{\mathrm{D}},s)$ and $F_2(r_{\mathrm{D}},s)$ are determined by Eqs. (14a) and (14b) in that we set $a_0 = 1, a_1 = 0$ and $a_0 = 0, a_1 = 1,$

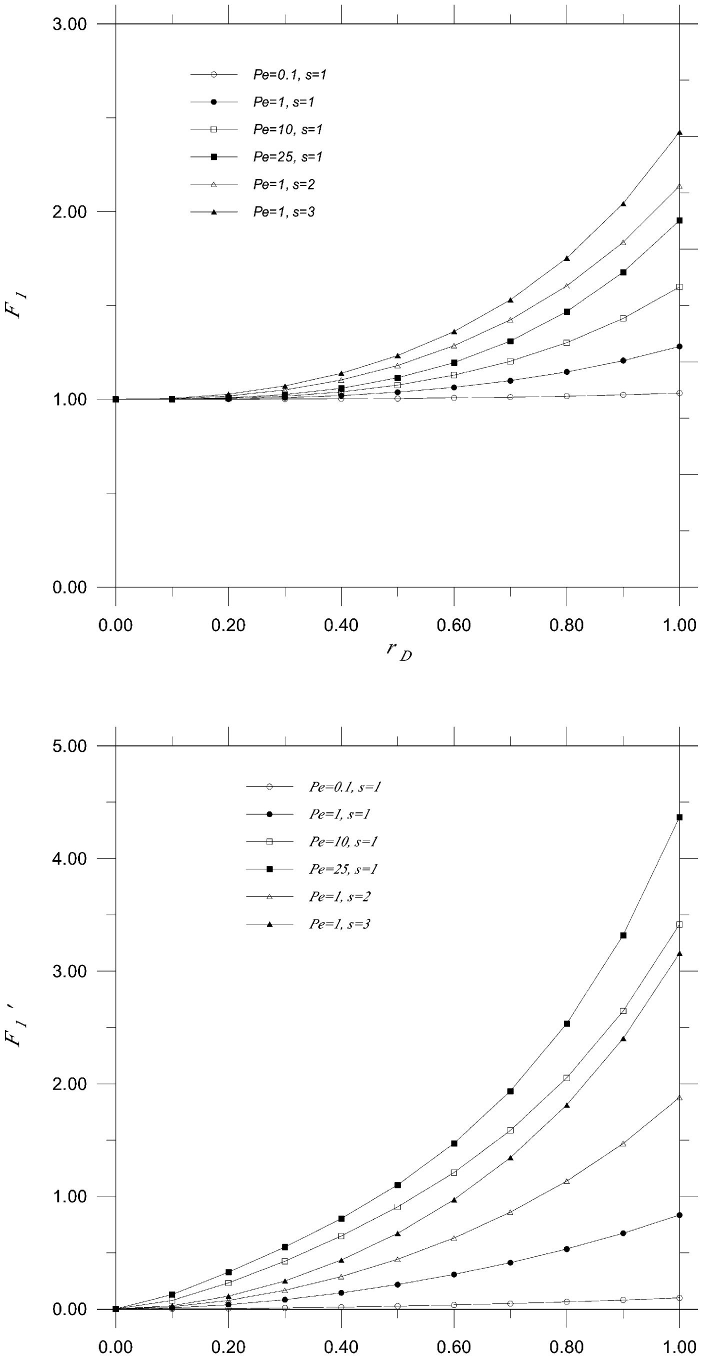

Fig. 5. Plot of (a) F_1 versus r_D for various Peclet numbers and *s*; (b) F_1' versus r_D for various Peclet numbers and *s*.

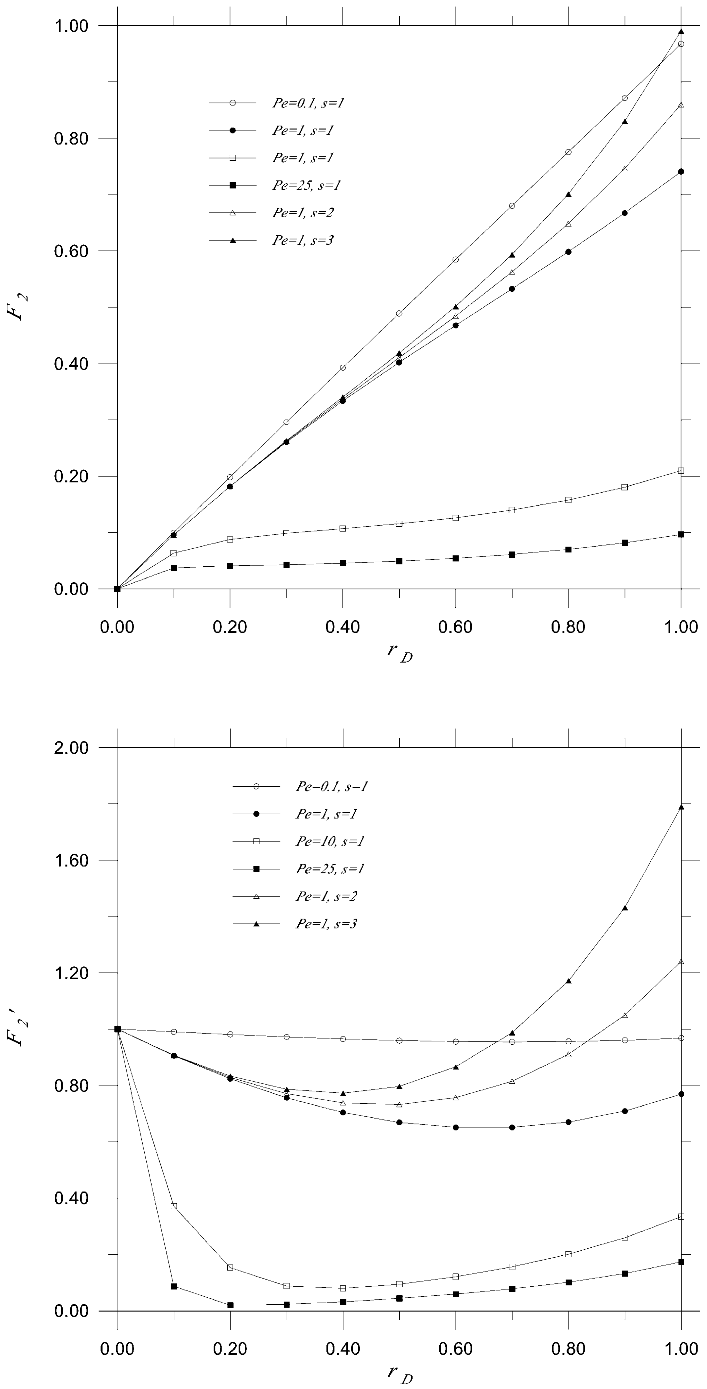

Fig. 6. Plot of (a) F_2 versus r_D for various Peclet numbers and s; (b) F_2' versus r_D for various Peclet numbers and s.

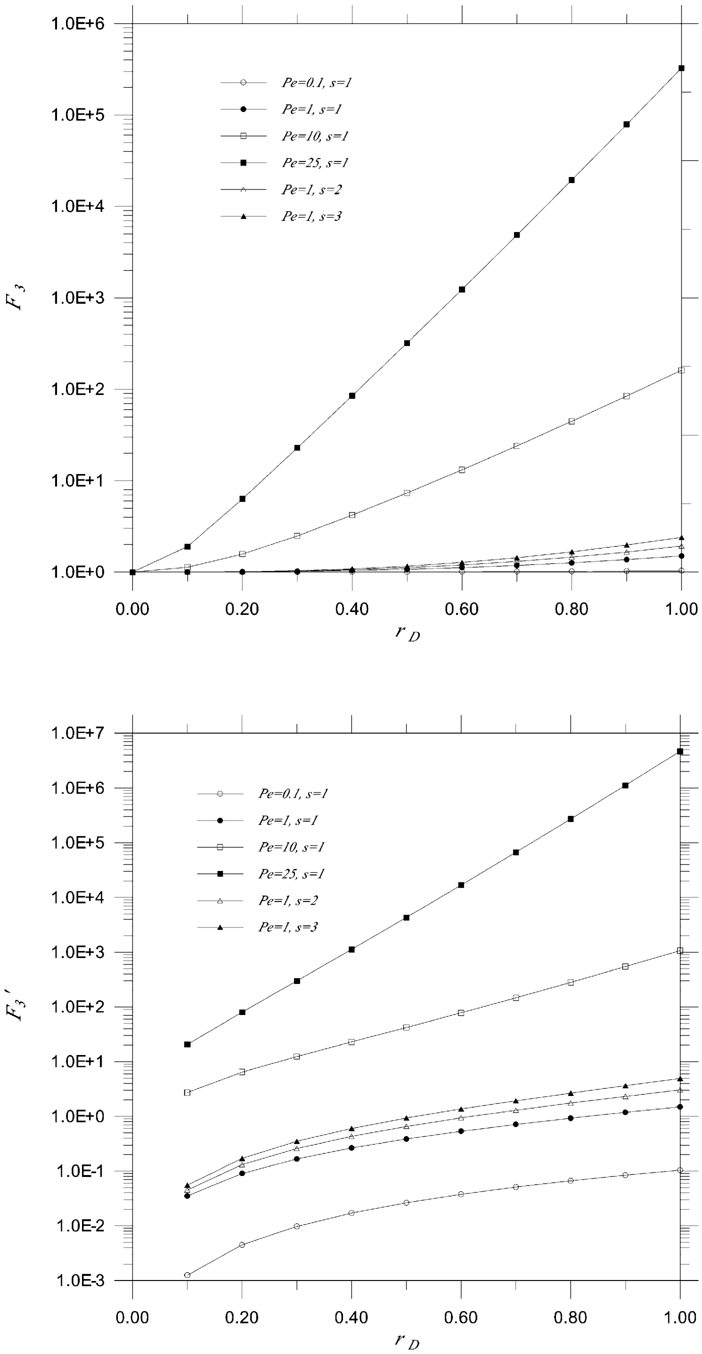

Fig. 7. Plot of (a) F_3 versus r_D for various Peclet numbers and *s*; (b) F'_3 versus r_D for various Peclet numbers and *s*.

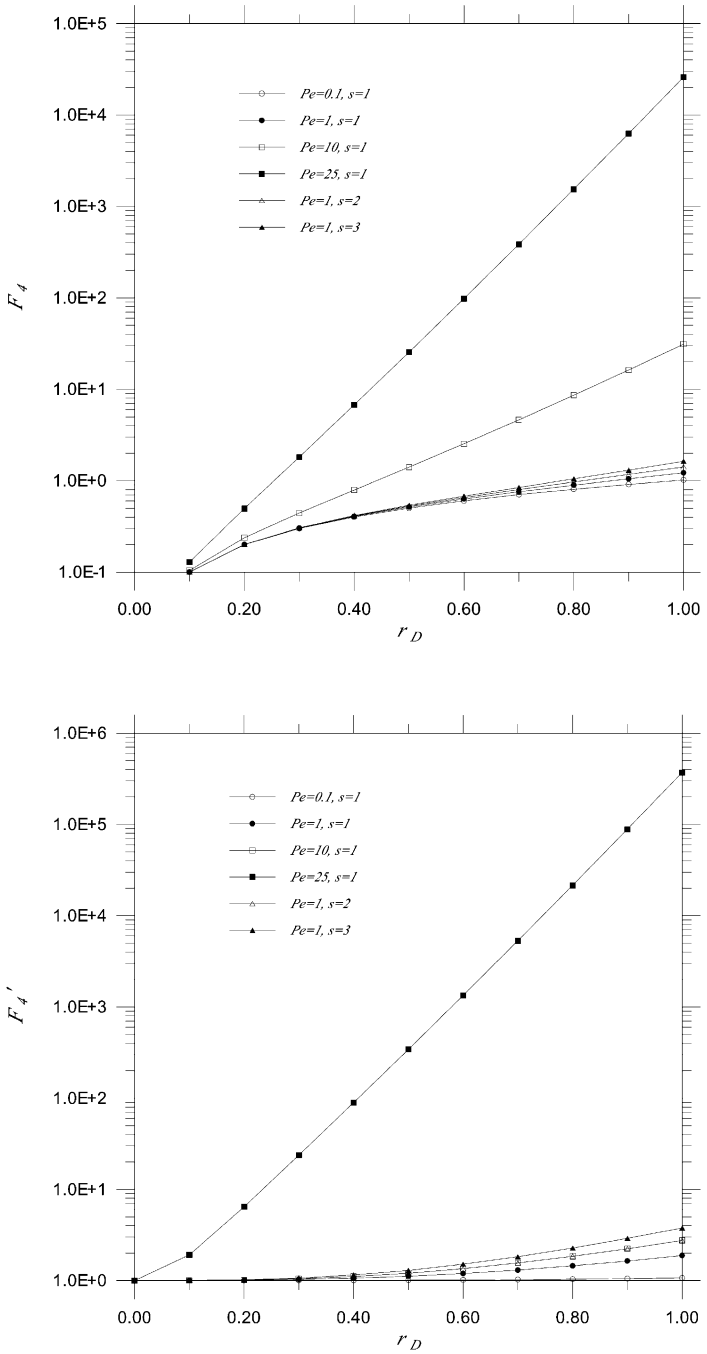

Fig. 8. Plot of (a) F_4 versus r_D for various Peclet numbers and s; (b) F'_4 versus r_D for various Peclet numbers and s.

respectively. The unknown coefficients, b_1 and b_2 of Eq. (15) are determined by the boundary conditions (6) and (7), to obtain a particular solution of Eq. (5).

A particular solution is obtained straightforwardly by imposing boundary conditions (6) and (7) on the general solution specified as Eq. (15). Therefore, the new Laplace transform power series solution at the pumping well may be expressed as

$$\bar{C}_D(r_{wD}, s) = \psi \frac{J(s)}{H(s)} \tag{16}$$

where

$$J(s) = (-AQ_{21} + CQ_{11})P_{11} + (AP_{21} - CP_{11})Q_{11};$$

$$H(s) = (AP_{21} - CP_{11})(AQ_{22} + DQ_{12}) - (AP_{22} + DP_{12})(AQ_{21} - CQ_{11})$$

$$A = \frac{1}{Pe}; \quad C = \mu_w s; \quad D = 1 + \mu_i s$$

$$P_{11} = F_1(r_{wD}, s); \quad P_{21} = \frac{\partial F_1(r_{wD}, s)}{\partial r_D};$$

$$P_{12} = F_1(1, s); \quad P_{22} = \frac{\partial F_1(1, s)}{\partial r_D}$$

$$Q_{11} = F_2(r_{wD}, s); \quad Q_{21} = \frac{\partial F_2(r_{wD}, s)}{\partial r_D};$$

$$Q_{12} = F_2(1, s); \quad Q_{22} = \frac{\partial F_2(1, s)}{\partial r_D}$$

3.2. Power series solution with variable change

The governing Eq. (5) involves the advection term that generally leads to a numerical error in numerical calculation for large Peclet numbers (Xu and Brusseau, 1995). Therefore, a variable change is used to eliminate the first derivative and convert Eq. (5) to

$$\frac{1}{Pe}\frac{d^2G}{dr_D^2} - \left(\frac{Pe}{4} + \frac{2r_D R}{1 - r_{wD}^2}s\right)G = 0 \tag{17}$$

where

$$G = \exp\left[\frac{Pe}{2}(1 - r_D)\right]\bar{C}_D$$

Expressing the boundary conditions (6) and (7) in terms of G,

$$\frac{1}{Pe}\frac{dG}{dr_D} - \left(\frac{1}{2} + \mu_w s\right)G = 0 \tag{18}$$

and

$$\frac{1}{Pe}\frac{dG}{dr_D} + \left(\frac{1}{2} + \mu_i s\right)G = 1 \tag{19}$$

The governing Eq. (17) is a self-adjoint operator which asserts a complete orthornormal set in Hilbert space and can be solved using the power series method (Gustafson, 1980) by substituting

$$G = \sum_{m=0}^{\infty} c_m r_D^m \tag{20}$$

and its first and second derivatives into Eq. (17), and imposing boundary conditions (18) and (19). Following the procedure outlined in Section 3.1 in shifting the summation indices and equating every power of r_D, we yield a recurrence relation among the coefficients of each power of r_D as follows.

$$c_2 = \frac{Pe}{8}c_0 \tag{21a}$$

$$c_{m+2} = \frac{Pe}{(m+2)(m+1)}\left(\frac{Pe}{4}c_m + \frac{2Rs}{1 - r_{wD}^2}c_{m-1}\right)$$

$$m = 1, 2, 3, \ldots \tag{21b}$$

The general solution to the governing Eq. (17) may be written as

$$G(r_D, s) = d_1 F_3(r_D, s) + d_2 F_4(r_D, s) \tag{22}$$

where $F_3(r_D, s)$ and $F_4(r_D, s)$ are two linearly independent general functions obtained by substituting the recurrence relations (21a) and (21b) into Eq. (20). Eqs. (21a) and (21b) determine the series coefficients of $F_3(r_D, s)$ and $F_4(r_D, s)$ in that we set $c_0 = 1$, $c_1 = 0$ and $c_0 = 0$, $c_1 = 1$, respectively.

Particular solutions can be uniquely determined by boundary conditions (18) and (19). Thus, the new Laplace transform power series solution with variable change at the pumping well can be written as

$$\bar{C}_{\rm D}(r_{\rm wD}, s) = \psi \exp\left[\frac{Pe}{2}(1 - r_{\rm wD})\right]\frac{K(s)}{L(s)} \quad (23)$$

where

$$K(s) = (-AT_{21} + CT_{11})R_{11} + (AR_{21} - CR_{11})T_{11};$$

$$L(s) = (AR_{21} - CR_{11})(AT_{22} + DT_{12}) - (AR_{22} + DR_{12})(AT_{21} - CT_{11});$$

$$A = \frac{1}{Pe};\ \ C = \tfrac{1}{2} + \mu_{\rm w}s;\ \ D = \tfrac{1}{2} + \mu_{\rm i}s;$$

$$R_{11} = F_3(r_{\rm wD}, s);\ \ R_{21} = \frac{\partial F_3(r_{\rm wD}, s)}{\partial r_{\rm D}};$$

$$R_{12} = F_3(1, s);\ \ R_{22} = \frac{\partial F_3(1, s)}{\partial r_{\rm D}};$$

$$T_{11} = F_4(r_{\rm wD}, s);\ \ T_{21} = \frac{\partial F_4(r_{\rm wD}, s)}{\partial r_{\rm D}};$$

$$T_{12} = F_4(1, s);\ \ T_{22} = \frac{\partial F_4(1, s)}{\partial r_{\rm D}}$$

The inverse Laplace transform of Eqs. (17) and (23) provides the temporal concentration at the pumping well. In this work, a numerical inverse Laplace transform is adopted to yield the solution. The de Hoog et al. (1982) algorithm is used to execute the numerical inversion because it is accurate for a wide range of functions and it also performs reasonably well in the neighborhood of a discontinuity (Moench, 1991). A FORTRAN subroutine, DINLAP/INLAP, provided by IMSL Subroutine Library (Visual Numerical Inc., 1994) and based on the de Hoog et al. algorithm, is employed to perform the numerical Laplace inversion.

4. Results and discussion

4.1. Validation of two power series solutions

The obtained solutions are compared with Moench's (1989) solution to demonstrate the accuracy of the developed Laplace transform direct power series solution and the power series solution with variable change. Figs. 1 and 2 plot curves at the pumping well for step and slug inputs of the tracer using various Peclet numbers, and compare those curves to Moench's (1989) solution. The pumping well mixing factor and injection well mixing factor are set to zero so that well bore mixing exerts no influence on breakthrough curves. Dimensionless concentrations obtained from the new Laplace transform direct power series method agree with those obtained from the Laplace special Airy function solution (Figs. 1 and 2). Fig. 3 depicts the effects of large well bore mixing of the concentration breakthrough curves for both solutions. The two solutions agree well with each other.

Only the direct power series solution and Moench's solution are compared. The power series solution with variable change gives exactly the same results as Moench's solution. Comparing the two developed solutions reveals that the new Laplace transform direct power series method and the power series method with variable change can accurately solve the radial advection–dispersion equations.

4.2. High Peclet numbers

The direct series method, however, did not yield the solution for large Peclet numbers when the rate of transport of the tracer by advection far exceeds that by dispersion. Such conditions cause breakthrough curves with a steep front. Transport in a single fracture or in a particularly homogeneous granular aquifer may involve large Peclet numbers (Moench, 1991). The power series method with variable change, however, can handle transport conditions of large Peclet numbers. Fig. 4 gives a comparison of the theoretical breakthrough curves from Moench's solution with that of the Laplace transform power series solution with variable change to eliminate the first derivative for a step tracer input using Peclet numbers of 50, 100, and 200. The two solutions agree excellently for large Peclet numbers. This result suggests that the new Laplace transform power series technique with variable change to eliminate the first spatial derivative, is robust and can be adopted to solve the radial advection–dispersion equation because it yields accurate solutions to radial dispersion problems that have large Peclet numbers and generally have a sharp front.

Both of the proposed Laplace transform power

Table A1
The dependence of the numbers of the required series term (m) for convergences of functions F_1, F'_1, F_2 and F'_2 ($Pe = 1$)

m	F_1		F'_1		F_2		F'_2	
	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum
0	1.00	1.00			0.00	0.00		
1	0.00	1.00	0.00	0.00	1.00	1.00	1.00	1.00
2	0.00	1.00	0.00	0.00	-5.00×10^{-1}	5.00×10^{-1}	-1.00	-1.49×10^{-8}
3	3.33×10^{-1}	1.33	1.00	1.00	1.67×10^{-1}	6.67×10^{-1}	5.00×10^{-1}	5.00×10^{-1}
4	-8.33×10^{-2}	1.25	-3.33×10^{-1}	6.67×10^{-1}	1.25×10^{-1}	7.92×10^{-1}	5.00×10^{-1}	1.00
5	1.67×10^{-2}	1.27	8.33×10^{-2}	7.50×10^{-1}	-7.50×10^{-2}	7.17×10^{-1}	-3.75×10^{-1}	6.25×10^{-1}
6	1.94×10^{-2}	1.29	1.17×10^{-1}	8.67×10^{-1}	2.36×10^{-2}	7.40×10^{-1}	1.42×10^{-1}	7.67×10^{-1}
7	-6.75×10^{-3}	1.28	-4.72×10^{-2}	8.19×10^{-1}	2.58×10^{-3}	7.43×10^{-1}	1.81×10^{-2}	7.85×10^{-1}
8	1.44×10^{-3}	1.28	1.15×10^{-2}	8.31×10^{-1}	-3.00×10^{-3}	7.40×10^{-1}	-2.40×10^{-2}	7.61×10^{-1}
9	3.80×10^{-4}	1.28	3.42×10^{-3}	8.34×10^{-1}	9.89×10^{-4}	7.41×10^{-1}	8.90×10^{-3}	7.70×10^{-1}
10	-1.88×10^{-4}	1.28	-1.88×10^{-3}	8.32×10^{-1}	-4.16×10^{-5}	7.41×10^{-1}	-4.16×10^{-4}	7.69×10^{-1}
11	4.32×10^{-5}	1.28	4.76×10^{-4}	8.33×10^{-1}	-5.08×10^{-5}	7.41×10^{-1}	-5.59×10^{-4}	7.69×10^{-1}
12	2.16×10^{-6}	1.28	2.59×10^{-5}	8.33×10^{-1}	1.92×10^{-5}	7.41×10^{-1}	2.31×10^{-4}	7.69×10^{-1}
13	-2.58×10^{-6}	1.28	-3.35×10^{-5}	8.33×10^{-1}	-2.01×10^{-6}	7.41×10^{-1}	-2.62×10^{-5}	7.69×10^{-1}
14	6.59×10^{-7}	1.28	9.23×10^{-6}	8.33×10^{-1}	-4.14×10^{-7}	7.41×10^{-1}	-5.80×10^{-6}	7.69×10^{-1}
15	-2.34×10^{-8}	1.28	-3.51×10^{-7}	8.33×10^{-1}	2.11×10^{-7}	7.41×10^{-1}	3.16×10^{-6}	7.69×10^{-1}
16	-2.00×10^{-8}	1.28	-3.20×10^{-7}	8.33×10^{-1}	-2.99×10^{-8}	7.41×10^{-1}	-4.79×10^{-7}	7.69×10^{-1}
17	6.02×10^{-9}	1.28	1.02×10^{-7}	8.33×10^{-1}	-1.29×10^{-9}	7.41×10^{-1}	-2.19×10^{-8}	7.69×10^{-1}
18	-4.87×10^{-10}	1.28	-8.77×10^{-9}	8.33×10^{-1}	1.45×10^{-9}	7.41×10^{-1}	2.61×10^{-8}	7.69×10^{-1}
19	-9.13×10^{-11}	1.28	-1.73×10^{-9}	8.33×10^{-1}	-2.51×10^{-10}	7.41×10^{-1}	-4.77×10^{-9}	7.69×10^{-1}
20	3.63×10^{-11}	1.28	7.25×10^{-10}	8.33×10^{-1}	5.80×10^{-12}	7.41×10^{-1}	1.16×10^{-10}	7.69×10^{-1}
21	-4.05×10^{-12}	1.28	-8.50×10^{-11}	8.33×10^{-1}	6.62×10^{-12}	7.41×10^{-1}	1.39×10^{-10}	7.69×10^{-1}
22	-2.11×10^{-13}	1.28	-4.65×10^{-12}	8.33×10^{-1}	-1.39×10^{-12}	7.41×10^{-1}	-3.06×10^{-11}	7.69×10^{-1}
23	1.53×10^{-13}	1.28	3.51×10^{-12}	8.33×10^{-1}	8.33×10^{-14}	7.41×10^{-1}	1.92×10^{-12}	7.69×10^{-1}
24	-2.10×10^{-14}	1.28	-5.05×10^{-13}	8.33×10^{-1}	2.05×10^{-14}	7.41×10^{-1}	4.92×10^{-13}	7.69×10^{-1}
25	1.37×10^{-16}	1.28	3.42×10^{-15}	8.33×10^{-1}	-5.45×10^{-15}	7.41×10^{-1}	-1.36×10^{-13}	7.69×10^{-1}
26			1.21×10^{-14}	8.33×10^{-1}	4.66×10^{-16}	7.41×10^{-1}	1.21×10^{-14}	7.69×10^{-1}
27			-2.08×10^{-15}	8.33×10^{-1}			1.11×10^{-15}	7.69×10^{-1}
28			8.72×10^{-17}	8.33×10^{-1}			-4.45×10^{-16}	7.69×10^{-1}

series techniques and the special Airy function solution can accurately handle the radial dispersion problems of large Peclet numbers. The proposed method, however, benefits from the fact that it is the standard approach to solving linear differential equations with variable coefficients and the solution procedures are straightforward in principle. Furthermore, the developed power series method with variable change does not suffer from the need to derive the analytical solution of the special functions in that the analytical solution is generally an unknown priori or unavailable. The proposed technique is parsimonious and easy to code into a program. The power series method with variable change was used to solve the one-dimensional radial advection–dispersion equation with variable-dependent coefficients. It is also valuable in modeling other hydrogeological problems with variable-dependent governing equations when the special function's analytical solutions are not available yet a continuous temporal and spatial solution is demanded. Consequently, the developed technique can be extended to solve higher-dimensional radial dispersion problems or a scale-dependent dispersion problem with variable dependent differential equation.

4.3. Mathematical behavior of power series functions

Two Laplace transform power series solutions, one with and one without the variable change include four new functions. These four functions are in the form of infinite series. Such an infinite series can be

Table A2
The dependence of the numbers of the required series term (m) for convergences of functions F_1, F_1', F_2 and F_2' ($Pe = 10$)

m	F_1		F_1'		F_2		F_2'	
	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum
0	1.00	1.00			0.00	0.00		
2	0.00	1.00	0.00	0.00	-5.00	-4.00	-1.00×10	-9.00
4	-8.33	-4.00	-3.33×10	-2.33×10	-4.00×10	-2.73×10	-1.60×10^{2}	-1.19×10^{2}
6	-2.56×10	-1.29×10	-1.53×10^{2}	-9.33×10	-1.14×10^{2}	-6.62×10	-6.83×10^{2}	-4.27×10^{2}
8	-3.47×10	-1.51×10	-2.78×10^{2}	-1.43×10^{2}	-1.53×10^{2}	-7.53×10	-1.22×10^{3}	-6.44×10^{2}
10	-2.43×10	-7.84	-2.43×10^{2}	-1.03×10^{2}	-1.06×10^{2}	-4.34×10	-1.06×10^{3}	-4.63×10^{2}
12	-8.34	-4.48×10^{-1}	-1.00×10^{2}	-2.95×10	-3.64×10	-1.10×10	-4.37×10^{2}	-1.43×10^{2}
14	-6.33×10^{-1}	2.23	-8.86	4.59	-2.71	6.22×10^{-1}	-3.79×10	5.88
16	5.08×10^{-1}	2.36	8.13	7.14	2.24	1.20	3.58×10	1.67×10
18	1.68×10^{-1}	2.18	3.02	4.28	7.32×10^{-1}	4.15×10^{-1}	1.32×10	4.18
20	1.11×10^{-2}	2.14	2.21×10^{-1}	3.39	4.77×10^{-2}	2.08×10^{-1}	9.54×10^{-1}	2.97×10^{-1}
22	-3.77×10^{-3}	2.13	-8.28×10^{-2}	3.37	-1.66×10^{-2}	2.03×10^{-1}	-3.64×10^{-1}	1.88×10^{-1}
24	-7.66×10^{-4}	2.14	-1.84×10^{-2}	3.40	-3.34×10^{-3}	2.09×10^{-1}	-8.03×10^{-2}	3.16×10^{-1}
26	-5.76×10^{-6}	2.14	-1.50×10^{-4}	3.40	-2.28×10^{-5}	2.10×10^{-1}	-5.93×10^{-4}	3.35×10^{-1}
28	1.18×10^{-5}	2.14	3.31×10^{-4}	3.40	5.18×10^{-5}	2.10×10^{-1}	1.45×10^{-3}	3.34×10^{-1}
30	9.53×10^{-7}	2.14	2.86×10^{-5}	3.40	4.15×10^{-6}	2.10×10^{-1}	1.24×10^{-4}	3.34×10^{-1}
32	-6.84×10^{-8}	2.14	-2.19×10^{-6}	3.40	-3.02×10^{-7}	2.10×10^{-1}	-9.66×10^{-6}	3.34×10^{-1}
34	-1.24×10^{-8}	2.14	-4.20×10^{-7}	3.40	-5.40×10^{-8}	2.10×10^{-1}	-1.83×10^{-6}	3.34×10^{-1}
36	-4.57×10^{-11}	2.14	-1.65×10^{-9}	3.40	-1.76×10^{-10}	2.10×10^{-1}	-6.35×10^{-9}	3.34×10^{-1}
38	7.94×10^{-11}	2.14	3.02×10^{-9}	3.40	3.48×10^{-10}	2.10×10^{-1}	1.32×10^{-8}	3.34×10^{-1}
40	3.03×10^{-12}	2.14	1.21×10^{-10}	3.40	1.31×10^{-11}	2.10×10^{-1}	5.25×10^{-10}	3.34×10^{-1}
42	-2.99×10^{-13}	2.14	-1.25×10^{-11}	3.40	-1.31×10^{-12}	2.10×10^{-1}	-5.51×10^{-11}	3.34×10^{-1}
44	-2.10×10^{-14}	2.14	-9.22×10^{-13}	3.40	-9.13×10^{-14}	2.10×10^{-1}	-4.02×10^{-12}	3.34×10^{-1}
46	6.38×10^{-16}	2.14	2.93×10^{-14}	3.40	2.82×10^{-15}	2.10×10^{-1}	1.30×10^{-13}	3.34×10^{-1}
48			3.99×10^{-15}	3.40	3.63×10^{-16}	2.10×10^{-1}	1.74×10^{-14}	3.34×10^{-1}
50							-8.78×10^{-17}	3.34×10^{-1}

straightforwardly evaluated. We have to consider, however, the behavior of the series that requires sufficient terms to be summed to obtain accurate results. The dependence of the required numbers of the series term on Peclet numbers provides insight into the convergence of two power series solutions. We have performed the computation to present the dependence of the required numbers of the series term on Peclet numbers (detailed computation results are provided in Appendix A). For a fixed tolerance error, the number of the required terms to be summed generally increase with increase of the Peclet number. For a fixed tolerance error of 10^{-6}, the required number of series terms for F_1 and F_2 in the direct power series method, is around 10, 34, and 280 for $Pe = 1$, 10, and 100, respectively; whereas for F_3 and F_4 in the power series method with variable change, the required number of series term is about 7, 16, and 80 for $Pe = 1$, 10, and 100, respectively. Comparing this convergence behavior of the evaluation of the developed functions with that of the generally used special Airy function yields interesting results. For the small Peclet number, the correspondingly small arguments of the Airy function, $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, are determined by (Abramowitz and Stegun, 1972, p. 446, Eqs. 10.4.2–5)

$$\mathrm{Ai}(z) = \beta_1 f(z) - \beta_2 g(z) \tag{24}$$

$$\mathrm{Bi}(z) = 3^{1/2}[\beta_1 f(z) + \beta_2 g(z)] \tag{25}$$

where

$$f(z) = 1 + \frac{1}{3!}z^3 + \frac{1\times4}{6!}z^6 + \frac{1\times4\times7}{9!}z^9 + \cdots \tag{26}$$

$$g(z) = z + \frac{2}{4!}z^3 + \frac{2\times5}{7!}z^7 + \frac{2\times5\times8}{10!}z^{10} + \cdots \tag{27}$$

Table A3
The dependence of the numbers of the required series term (m) for convergences of functions F_1, F'_1, F_2 and F'_2 ($Pe = 100$)

m	F_1		F'_1		F_2		F'_2	
	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum
0	1.00	1.00			0.00	0.00		
10	-5.48×10^{9}	-4.98×10^{9}	-5.48×10^{10}	-5.02×10^{10}	-2.74×10^{11}	-2.49×10^{11}	-2.74×10^{12}	-2.51×10^{12}
20	-7.95×10^{17}	-6.61×10^{17}	-1.59×10^{19}	-1.33×10^{19}	-3.97×10^{19}	-3.30×10^{19}	-7.94×10^{20}	-6.66×10^{20}
30	-6.96×10^{23}	-5.33×10^{23}	-2.09×10^{25}	-1.61×10^{25}	-3.47×10^{25}	-2.66×10^{25}	-1.04×10^{27}	-8.05×10^{26}
40	-2.11×10^{28}	-1.50×10^{28}	-8.45×10^{29}	-6.05×10^{29}	-1.06×10^{30}	-7.50×10^{29}	-4.22×10^{31}	-3.02×10^{31}
50	-5.18×10^{31}	-3.44×10^{31}	-2.59×10^{33}	-1.73×10^{33}	-2.59×10^{33}	-1.72×10^{33}	-1.29×10^{35}	-8.64×10^{34}
60	-1.70×10^{34}	-1.05×10^{34}	-1.02×10^{36}	-6.36×10^{35}	-8.48×10^{35}	-5.26×10^{35}	-5.09×10^{37}	-3.18×10^{37}
70	-1.03×10^{36}	-6.03×10^{35}	-7.24×10^{37}	-4.25×10^{37}	-5.17×10^{37}	-3.01×10^{37}	-3.62×10^{39}	-2.12×10^{39}
80	-1.49×10^{37}	-8.18×10^{36}	-1.19×10^{39}	-6.58×10^{38}	-7.42×10^{38}	-4.08×10^{38}	-5.94×10^{40}	-3.29×10^{40}
90	-6.01×10^{37}	-3.13×10^{37}	-5.41×10^{39}	-2.83×10^{39}	-3.00×10^{39}	-1.56×10^{39}	-2.70×10^{41}	-1.41×10^{41}
100	-7.87×10^{37}	-3.88×10^{37}	-7.87×10^{39}	-3.90×10^{39}	-3.93×10^{39}	-1.94×10^{39}	-3.93×10^{41}	-1.95×10^{41}
110	-3.72×10^{37}	-1.74×10^{37}	-4.09×10^{39}	-1.93×10^{39}	-1.86×10^{39}	-8.71×10^{38}	-2.04×10^{41}	-9.63×10^{40}
120	-6.93×10^{36}	-3.10×10^{36}	-8.32×10^{38}	-3.74×10^{38}	-3.46×10^{38}	-1.55×10^{38}	-4.16×10^{40}	-1.87×10^{40}
130	-5.51×10^{35}	-2.35×10^{35}	-7.16×10^{37}	-3.07×10^{37}	-2.75×10^{37}	-1.17×10^{37}	-3.58×10^{39}	-1.53×10^{39}
140	-1.99×10^{34}	-8.12×10^{33}	-2.78×10^{36}	-1.14×10^{36}	-9.93×10^{35}	-4.05×10^{35}	-1.39×10^{38}	-5.70×10^{37}
150	-3.43×10^{32}	-1.34×10^{32}	-5.15×10^{34}	-2.02×10^{34}	-1.72×10^{34}	-6.71×10^{33}	-2.57×10^{36}	-1.01×10^{36}
160	-2.98×10^{30}	-1.12×10^{30}	-4.77×10^{32}	-1.80×10^{32}	-1.49×10^{32}	-5.59×10^{31}	-2.38×10^{34}	-8.98×10^{33}
170	-1.36×10^{28}	-4.89×10^{27}	-2.30×10^{30}	-8.35×10^{29}	-6.77×10^{29}	-2.44×10^{29}	-1.15×10^{32}	-4.17×10^{31}
180	-3.34×10^{25}	-1.16×10^{25}	-6.02×10^{27}	-2.10×10^{27}	-1.67×10^{27}	-5.81×10^{26}	-3.01×10^{29}	-1.05×10^{29}
190	-4.62×10^{22}	-1.63×10^{22}	-8.78×10^{24}	-5.25×10^{24}	-2.31×10^{24}	-1.43×10^{24}	-4.39×10^{26}	-2.30×10^{26}
200	-3.69×10^{19}	-8.34×10^{20}	-7.38×10^{21}	-2.30×10^{24}	-1.84×10^{21}	-6.57×10^{23}	-3.69×10^{23}	-8.27×10^{25}
210	-1.74×10^{16}	-8.22×10^{20}	-3.66×10^{18}	-2.30×10^{24}	-8.71×10^{17}	-6.56×10^{23}	-1.83×10^{20}	-8.26×10^{25}
220	-4.99×10^{12}	-8.22×10^{20}	-1.10×10^{15}	-2.30×10^{24}	-2.49×10^{14}	-6.56×10^{23}	-5.49×10^{16}	-8.26×10^{25}
230	-8.85×10^{8}	-8.22×10^{20}	-2.04×10^{11}	-2.30×10^{24}	-4.42×10^{10}	-6.56×10^{23}	-1.02×10^{13}	-8.26×10^{25}
240	-9.90×10^{4}	-8.22×10^{20}	-2.38×10^{7}	-2.30×10^{24}	-4.95×10^{6}	-6.56×10^{23}	-1.19×10^{9}	-8.26×10^{25}
250	-7.12	-8.22×10^{20}	-1.78×10^{3}	-2.30×10^{24}	-3.56×10^{2}	-6.56×10^{23}	-8.89×10^{4}	-8.26×10^{25}
260	-3.34×10^{-4}	-8.22×10^{20}	-8.69×10^{-2}	-2.30×10^{24}	-1.67×10^{-2}	-6.56×10^{23}	-4.34	-8.26×10^{25}
270	-1.04×10^{-8}	-8.22×10^{20}	-2.81×10^{-6}	-2.30×10^{24}	-5.20×10^{-7}	-6.56×10^{23}	-1.40×10^{-4}	-8.26×10^{25}
280	-2.18×10^{-13}	-8.22×10^{20}	-6.09×10^{-11}	-2.30×10^{24}	-1.09×10^{-11}	-6.56×10^{23}	-3.04×10^{-9}	-8.26×10^{25}
290			-8.98×10^{-16}	-2.30×10^{24}			-4.49×10^{-14}	-8.26×10^{25}

$\beta_1 = 3^{-2/3}/\Gamma(2/3) = 0.3550285\cdots$, $\beta_2 = 3^{-1/3}/\Gamma(1/3) = 0.25881940\cdots$, where $\Gamma(\bullet)$ is the gamma function. As posited by Hsieh (1986), 18 terms for both $f(z)$ and $g(z)$ are required sufficiently to approximate Eqs. (26) and (27) accurately. The asymptotic expansion can be used when large Peclet numbers result in large argument of the Airy function, (Abramowitz and Stegun, 1964, p. 449, Eq. 10.4.63):

$$\mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4}\, \mathrm{e}^{-\xi} \sum_{k=0}^{\infty} (-1)^k a_k \xi^{-k} \tag{28}$$

$$\mathrm{Bi}(z) = \pi^{-1/2} z^{-1/4}\, \mathrm{e}^{-\xi} \sum_{k=0}^{\infty} (-1)^k a_k \xi^{-k} \tag{29}$$

where $\xi = (2/3)z^{3/2}$, and the coefficients a_k are given by

$$a_k = \frac{(2k+1)(2k+3)\cdots(6k-1)}{216^k k!} \tag{30}$$

Hsieh (1986) stated that 14 terms are required sufficiently for computating Eq. (30) accurately. The developed functions clearly require more terms than the Airy function for large Peclet number to obtain sufficiently accurate calculation. The convergence of the series can also seemingly be further improved by developing other series asymptotic expansion methods, similar to that of Eqs. (28) and (29), for large Peclet numbers. Efforts to obtain a series approximation involve a complicated asymptotic series expansion and are suggested for further study.

Table A4
The dependence of the numbers of the required series term (m) for convergences of function F_3, F'_3, F_4 and F'_4 ($Pe = 1$)

m	F_3		F'_3		F_4		F'_4	
	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum
0	1.00	1.00			0.00	0.00		
1	0.00	1.00	0.00	0.00	1.00	1.00	1.00	1.00
2	1.25×10^{-1}	1.13	2.50×10^{-1}	2.50×10^{-1}	0.00	1.00	0.00	1.00
3	3.33×10^{-1}	1.46	1.00	1.25	4.17×10^{-2}	1.04	1.25×10^{-1}	1.13
4	2.60×10^{-3}	1.46	1.04×10^{-2}	1.26	1.67×10^{-1}	1.21	6.67×10^{-1}	1.79
5	1.67×10^{-2}	1.48	8.33×10^{-2}	1.34	5.21×10^{-4}	1.21	2.60×10^{-3}	1.79
6	2.22×10^{-2}	1.50	1.33×10^{-1}	1.48	4.17×10^{-3}	1.21	2.50×10^{-2}	1.82
7	2.23×10^{-4}	1.50	1.56×10^{-3}	1.48	7.94×10^{-3}	1.22	5.56×10^{-2}	1.87
8	6.95×10^{-4}	1.50	5.56×10^{-3}	1.48	3.72×10^{-5}	1.22	2.98×10^{-4}	1.88
9	6.19×10^{-4}	1.50	5.57×10^{-3}	1.49	1.43×10^{-4}	1.22	1.29×10^{-3}	1.88
10	6.89×10^{-6}	1.50	6.89×10^{-5}	1.49	1.77×10^{-4}	1.22	1.77×10^{-3}	1.88
11	1.40×10^{-5}	1.50	1.54×10^{-4}	1.49	1.00×10^{-6}	1.22	1.10×10^{-5}	1.88
12	9.39×10^{-6}	1.50	1.13×10^{-4}	1.49	2.51×10^{-6}	1.22	3.01×10^{-5}	1.88
13	1.11×10^{-7}	1.50	1.44×10^{-6}	1.49	2.26×10^{-6}	1.22	2.94×10^{-5}	1.88
14	1.67×10^{-7}	1.50	2.34×10^{-6}	1.49	1.45×10^{-8}	1.22	2.02×10^{-7}	1.88
15	8.95×10^{-8}	1.50	1.34×10^{-6}	1.49	2.66×10^{-8}	1.22	3.98×10^{-7}	1.88
16	1.10×10^{-9}	1.50	1.76×10^{-8}	1.49	1.89×10^{-8}	1.22	3.02×10^{-7}	1.88
17	1.31×10^{-9}	1.50	2.23×10^{-8}	1.49	1.31×10^{-10}	1.22	2.22×10^{-9}	1.88
18	5.86×10^{-10}	1.50	1.05×10^{-8}	1.49	1.89×10^{-10}	1.22	3.40×10^{-9}	1.88
19	7.38×10^{-12}	1.50	1.40×10^{-10}	1.49	1.11×10^{-10}	1.22	2.10×10^{-9}	1.88
20	7.29×10^{-12}	1.50	1.46×10^{-10}	1.49	8.12×10^{-13}	1.22	1.62×10^{-11}	1.88
21	2.80×10^{-12}	1.50	5.87×10^{-11}	1.49	9.66×10^{-13}	1.22	2.03×10^{-11}	1.88
22	3.59×10^{-14}	1.50	7.89×10^{-13}	1.49	4.79×10^{-13}	1.22	1.05×10^{-11}	1.88
23	3.02×10^{-14}	1.50	6.94×10^{-13}	1.49	3.69×10^{-15}	1.22	8.48×10^{-14}	1.88
24	1.01×10^{-14}	1.50	2.43×10^{-13}	1.49	3.72×10^{-15}	1.22	8.92×10^{-14}	1.88
25	1.32×10^{-16}	1.50	3.30×10^{-15}	1.49	1.60×10^{-15}	1.22	4.00×10^{-14}	1.88
26			2.52×10^{-15}	1.49	1.28×10^{-17}	1.22	3.32×10^{-16}	1.88
27			7.82×10^{-16}	1.49				

Notably, when evaluating the direct power series solution, some terms of the series F_1, F_2 and their derivatives exceed 10^{37} for a Peclet number of 100. The final sums of the functions, however, are less than 10^{21}. If the machine's precision is insufficient to cover the large numerical range of the arithmetic operation, the final sum may be inaccurate owing to the round-off error. This finding suggests that a high precision computer is required to ensure the convergence of the power series and to preserve the accuracy of the solution. The mathematical characteristics of the four new functions are examined closely. Figs. 5–8(a) and (b) plot the four functions, $F_1(r_D, s)$, $F_2(r_D, s)$, $F_3(r_D, s)$ and $F_4(r_D, s)$, as well as their first derivatives versus the increase of r_D for various Pe and s. The four functions and their first derivatives all increase with the increase of r_D as r_D approaches 1. The mathematical behaviors of the four functions and their derivative are fundamentally different from that of the special Airy function because $\mathrm{Ai}(z)$ decreases as z increases. The proposed method is suggested as more appropriately applicable to problems with a finite domain because neither F_1 and F_3 nor F_2 and F_4 decreases with increasing r_D for large r_D. Conversely, the Airy function is most applicable to the problem with an infinite domain because $\mathrm{Ai}(z)$ decreases to zero as z approaches infinity, whereas $\mathrm{Bi}(z)$ is unbounded as z approaches infinity. The coefficient of $\mathrm{Bi}(z)$ then can be cancelled out as the infinite boundary condition is used to determine the unknown coefficients of the general solution.

Table A5
The dependence of the numbers of the required series term (m) for convergences of functions F_3, F_3', F_4 and F_4' ($Pe = 10$)

m	F_3		F_3'		F_4		F_4'	
	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum
0	1.00	1.00			0.00	0.00		
2	1.25×10	1.35×10	2.50×10	2.50×10	0.00	1.00	0.00	1.00
4	2.60×10	4.29×10	1.04×10^2	1.39×10^2	1.67	6.83	6.67	2.02×10
6	2.39×10	8.35×10	1.44×10^2	3.66×10^2	4.17	1.62×10	2.50×10	7.12×10
8	1.66×10	1.22×10^2	1.33×10^2	6.55×10^2	3.72	2.38×10	2.98×10	1.28×10^2
10	9.58	1.46×10^2	9.58×10	8.81×10^2	1.90	2.82×10	1.90×10	1.70×10^2
12	4.00	1.57×10^2	4.79×10	9.98×10^2	7.40×10^{-1}	3.02×10	8.88	1.92×10^2
14	1.24	1.60×10^2	1.74×10	1.04×10^3	2.39×10^{-1}	3.09×10	3.34	2.01×10^2
16	3.16×10^{-1}	1.61×10^2	5.05	1.06×10^3	6.18×10^{-2}	3.11×10	9.89×10^{-1}	2.04×10^2
18	6.81×10^{-2}	1.61×10^2	1.23	1.06×10^3	1.31×10^{-2}	3.11×10	2.36×10^{-1}	2.05×10^2
20	1.24×10^{-2}	1.61×10^2	2.48×10^{-1}	1.06×10^3	2.38×10^{-3}	3.11×10	4.76×10^{-2}	2.05×10^2
22	1.95×10^{-3}	1.61×10^2	4.28×10^{-2}	1.06×10^3	3.77×10^{-4}	3.11×10	8.29×10^{-3}	2.05×10^2
24	2.69×10^{-4}	1.61×10^2	6.46×10^{-3}	1.06×10^3	5.20×10^{-5}	3.11×10	1.25×10^{-3}	2.05×10^2
26	3.30×10^{-5}	1.61×10^2	8.59×10^{-4}	1.06×10^3	6.36×10^{-6}	3.11×10	1.65×10^{-4}	2.05×10^2
28	3.62×10^{-6}	1.61×10^2	1.01×10^{-4}	1.06×10^3	6.99×10^{-7}	3.11×10	1.96×10^{-5}	2.05×10^2
30	3.59×10^{-7}	1.61×10^2	1.08×10^{-5}	1.06×10^3	6.93×10^{-8}	3.11×10	2.08×10^{-6}	2.05×10^2
32	3.23×10^{-8}	1.61×10^2	1.03×10^{-6}	1.06×10^3	6.23×10^{-9}	3.11×10	1.99×10^{-7}	2.05×10^2
34	2.66×10^{-9}	1.61×10^2	9.05×10^{-8}	1.06×10^3	5.13×10^{-10}	3.11×10	1.75×10^{-8}	2.05×10^2
36	2.02×10^{-10}	1.61×10^2	7.26×10^{-9}	1.06×10^3	3.89×10^{-11}	3.11×10	1.40×10^{-9}	2.05×10^2
38	1.41×10^{-11}	1.61×10^2	5.36×10^{-10}	1.06×10^3	2.72×10^{-12}	3.11×10	1.03×10^{-10}	2.05×10^2
40	9.17×10^{-13}	1.61×10^2	3.67×10^{-11}	1.06×10^3	1.77×10^{-13}	3.11×10	7.07×10^{-12}	2.05×10^2
42	5.55×10^{-14}	1.61×10^2	2.33×10^{-12}	1.06×10^3	1.07×10^{-14}	3.11×10	4.49×10^{-13}	2.05×10^2
44	3.14×10^{-15}	1.61×10^2	1.38×10^{-13}	1.06×10^3	6.05×10^{-16}	3.11×10	2.66×10^{-14}	2.05×10^2
46			7.65×10^{-15}	1.06×10^3			1.48×10^{-15}	2.05×10^2
48			3.98×10^{-16}	1.06×10^3				

5. Conclusion

This study presents a novel Laplace transform power series method to solve the variable-dependent radial advection–dispersion differential equation. Of the two power series methods presented, one is directly applied to solve the transformed ordinary differential equation, whereas the other employs variable change to eliminate the first spatial derivative and applies the power series method to solve the transformed ordinary differential equation. The new solutions are compared with Moench's solution that employs the special Airy function to solve the transform equations. Results show that the new Laplace transform power series technique with variable change to eliminate the first spatial derivative, provides an accurate and robust solution of the radial advection–dispersion equation that describes solute transport in porous media in a radially convergent tracer test. The direct power series method, however, does not permit evaluation of the solution for large Peclet numbers, unless the machine's precision is sufficiently to cover the large numerical range of the arithmetic operation. The novel power series technique with variable change approach is suggested to solve the radial advection–dispersion problems over a wide range of Peclet number. Furthermore, the novel power series technique with variable change can be extended to higher dimensional hydrogeological issues that have no analytical solution yet a temporal and spatial solution is demanded.

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Table A6
The dependence of the numbers of the required series term (m) for convergences of functions F_3, F_3', F_4 and F_4' ($Pe = 100$)

m	F_3		F_3'		F_4		F_4'	
	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum	Coeff.	Sum
0	1.00	1.00			0.00	0.00		
10	2.69×10^{10}	2.80×10^{10}	2.69×10^{11}	2.78×10^{11}	1.72×10^{7}	1.28×10^{8}	1.72×10^{8}	1.17×10^{9}
20	3.95×10^{15}	4.86×10^{15}	7.90×10^{16}	9.53×10^{16}	1.13×10^{13}	4.92×10^{13}	2.26×10^{14}	9.33×10^{14}
30	3.68×10^{18}	6.35×10^{18}	1.10×10^{20}	1.84×10^{20}	2.38×10^{16}	9.63×10^{16}	7.15×10^{17}	2.75×10^{18}
40	1.30×10^{20}	3.91×10^{20}	5.18×10^{21}	1.48×10^{22}	1.41×10^{18}	7.17×10^{18}	5.64×10^{19}	2.69×10^{20}
50	4.18×10^{20}	2.97×10^{21}	2.09×10^{22}	1.35×10^{23}	6.23×10^{18}	5.83×10^{19}	3.11×10^{20}	2.64×10^{21}
60	2.06×10^{20}	5.99×10^{21}	1.23×10^{22}	3.01×10^{23}	3.64×10^{18}	1.20×10^{20}	2.19×10^{20}	6.00×10^{21}
70	2.14×10^{19}	6.77×10^{21}	1.50×10^{21}	3.51×10^{23}	4.10×10^{17}	1.35×10^{20}	2.87×10^{19}	7.01×10^{21}
80	5.88×10^{17}	6.83×10^{21}	4.70×10^{19}	3.54×10^{23}	1.16×10^{16}	1.36×10^{20}	9.28×10^{17}	7.09×10^{21}
90	5.00×10^{15}	6.83×10^{21}	4.50×10^{17}	3.55×10^{23}	9.96×10^{13}	1.36×10^{20}	8.96×10^{15}	7.09×10^{21}
100	1.50×10^{13}	6.83×10^{21}	1.50×10^{15}	3.55×10^{23}	3.00×10^{11}	1.36×10^{20}	3.00×10^{13}	7.09×10^{21}
110	1.77×10^{10}	6.83×10^{21}	1.95×10^{12}	3.55×10^{23}	3.53×10^{8}	1.36×10^{20}	3.89×10^{10}	7.09×10^{21}
120	8.94×10^{6}	6.83×10^{21}	1.07×10^{9}	3.55×10^{23}	1.79×10^{5}	1.36×10^{20}	2.15×10^{7}	7.09×10^{21}
130	2.09×10^{3}	6.83×10^{21}	2.72×10^{5}	3.55×10^{23}	4.18×10	1.36×10^{20}	5.43×10^{3}	7.09×10^{21}
140	2.40×10^{-1}	6.83×10^{21}	3.36×10	3.55×10^{23}	4.80×10^{-3}	1.36×10^{20}	6.72×10^{-1}	7.09×10^{21}
150	1.43×10^{-5}	6.83×10^{21}	2.15×10^{-3}	3.55×10^{23}	2.86×10^{-7}	1.36×10^{20}	4.29×10^{-5}	7.09×10^{21}
160	4.63×10^{-10}	6.83×10^{21}	7.41×10^{-8}	3.55×10^{23}	9.26×10^{-12}	1.36×10^{20}	1.48×10^{-9}	7.09×10^{21}
170	8.48×10^{-15}	6.83×10^{21}	1.44×10^{-12}	3.55×10^{23}			2.88×10^{-14}	7.09×10^{21}

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Appendix A. Appendix

Tables A1–A6 present the computational results of the dependence of the required numbers of the series term for $r_D = 1$ and for various Peclet numbers, and specify power coefficients and partial sums for functions F_1, F_2, F_3, and F_4 and their first derivatives.

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