§0 heat equation

(M, g) : compact, connected, oriented, $\partial M = \phi$

$\Rightarrow \exists\, k(t, x, y)$: fundamental solution to the heat equation

- $\left(\frac{\partial}{\partial t} - \Delta_x\right) k = 0$
- $k(t, x, y) = k(t, y, x)$
- $\int_{y \in M} k(t, x, y) f(y)\, d\mu_y \longrightarrow f(x)$ as $t \to 0$
- $k(t, x, y) = \sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$

Solve $\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) u = v(t, x) \\ u(0, x) = u_0(x) \end{cases}$ given

recall $\left(\frac{\partial}{\partial t} + \lambda\right) u(t) = v(t)$

$\Rightarrow \frac{\partial}{\partial t}\left(e^{t\lambda} u\right) = e^{t\lambda} v(t)$

$\Rightarrow e^{t\lambda} u(t) - u(0) = \int_0^t e^{s\lambda} v(s)\, ds$

$\Rightarrow u(t) = e^{-t\lambda} u(0) + \int_0^t e^{(t-s)\lambda} v(s)\, ds$

$$u(t, x) = \int_{y \in M} k(t, x, y) u_0(y)\, d\mu_y + \int_0^t \int_{y \in M} k(t-s, x, y) v(s, y)\, d\mu_y\, ds$$

It can be verified by a direct computation for smooth $u_0(x)$ and $v(t,x)$

<u>defn</u> $Z(t) = \sum_{j\geq 0} e^{-\lambda_j t}$ is called the partition function of (M,g)

(or $Z(t;M,g)$)

Note that $Z(t) = \int_{x\in M} k(t,x,x)\, d\mu_x$

<u>rmk</u> • $Z(t)$ is equivalent to the data $\{\lambda_i\}_{i\geq 0}$

• $k(t,x,x) = \frac{1}{(4\pi t)^{n/2}} 1 \cdot (\underbrace{u_0(x,x)}_{=1} + O(t))$ as $t\to 0$

$\Rightarrow Z(t) \sim \frac{1}{(4\pi t)^{n/2}} Vol(M,g)$

§I. isoperimetric estimator and heat kernel

Suppose that (M,g) admits an isoperimetric estimator $H(\beta)$ with $H(\beta)\sim c\beta^\alpha$ for $\beta\sim 0$

$(\alpha\in[\frac{n-1}{n}, 1))$

(and $\sim c(1-\beta)^\alpha$ for $\beta\sim 1$)

$M^* = S^{n-1}\times(0,L)\cup N\cup S$

$\int_0^1 \frac{1}{\beta}d\beta = \infty$

(with a choice of $V^* = Vol(M^*,g^*)$)

<u>theorem</u> Denote by $k_*(t, N, N)$ the heat kernel (assume its existence). Then,

$$\begin{aligned} Z(t) &\le \mathrm{Vol}(M, g) \sup_x k(t,x,x) \\ &\le \mathrm{Vol}(M^*, g^*)\, k_*(t, N, N) \end{aligned}$$

<u>sketch</u> Solve $\begin{cases} (\frac{\partial}{\partial t} - \Delta) u = 0 \\ u(0,x) = u_0(x) \end{cases}$

and the "symmetrized one" on (M^*, g^*)

As $u_0(x) \to \delta_{x_0}(x)$ the Dirac measure

$\Rightarrow$ obtain the comparison at some $x_0 \in M$

$$\left(\int k(t,x,y)\, u_0(y)\, dy \longrightarrow k(t, x, x_0) \right)$$

<u>step 1</u> $u : M \to \mathbb{R}_{>0}$ smooth

<u>goal</u> construct 1D comparison datum depending on $\beta \in [0,1]$

$\mathrm{Range}(u) \subset [0, c = \sup u]$

Let $D(r) = \{ x \in M : u(x) > r \}$

$b(r) = \mathrm{Vol}(D(r)) / \mathrm{Vol}(M)$ $\searrow$ in r

$\Omega(r) = \{ u > r \}$

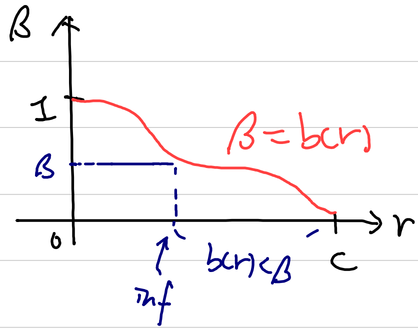
$b(r)$ = ratio of volume

$\bar{u} : [0,1] \to (0, C]$

ideally $\beta \in [0,1] \Rightarrow \beta = b(r)$ for some $r \in (0, C]$

Define $\bar{u}(\beta) =$ that r

$\Rightarrow \bar{u}(\beta) = \inf \{ r : b(r) < \beta \}$



$\Rightarrow \bar{u}(b(r)) = r \quad (a.e.)$

$\bar{u}$: { volume ratio of sub-level set of u } $\longrightarrow$ value of u

Now, let $\Omega(\beta) = D(\bar{u}(\beta)) = \{ u(x) > \bar{u}(\beta) \}$

Consider $F(\beta_0) = \int_{\Omega(\beta_0)} u(x)\, d\mu_x$ as $\Omega(b(r)) = D(r)$

$$= \int_{\bar{u}(\beta_0)}^{C} \left(\int_{\partial\Omega(b(r))} u\, |\nabla u|^{-1} \right) dr$$

$$= \int_{\bar{u}(\beta_0)}^{C} \left(r \int_{\partial\Omega(b(r))} |\nabla u|^{-1} \right) dr$$

$$\mathrm{Vol}(M)\, b(r) = \mathrm{Vol}(D(r))$$

$$= \int_r^C \left(\int_{\partial D(\rho)} |\nabla u|^{-1} \right) d\rho$$

$$\Rightarrow \mathrm{Vol}(M)\, b'(r) = - \int_{\partial D(r)} |\nabla u|^{-1}$$

$$F(\beta_0) = \int_{\bar{u}(\beta_0)}^{C} \left(-r\, \mathrm{Vol}(M)\, b'(r)\right) dr$$

Use $\beta = b(r)$ $\quad d\beta = b'(r)\, dr$

$r \in [\bar{u}(\beta_0), C] \iff \beta \in [0, \beta_0]$

and $r = \bar{u}(\beta)$

$$F(\beta_0) = \mathrm{Vol}(M) \int_0^{\beta_0} \bar{u}(\beta)\, d\beta$$

Relate $\int_{\Omega(\beta)} \Delta u$ to $F''(\beta)$:

- $F'(\beta) = \mathrm{Vol}(M)\, \bar{u}(\beta)$

$$\bar{u}(b(r)) = r \Rightarrow \bar{u}'(\beta) = \frac{1}{u'(r)} = \frac{1}{b'(\bar{u}(\beta))}$$

$$\Rightarrow F''(\beta) = \mathrm{Vol}(M) \Big/ b'(\bar{f}(\beta))$$

- $\displaystyle\int_{\Omega(\beta)} \Delta u \, d\mu_x = \int_{\partial\Omega(\beta)} \underbrace{\frac{\partial u}{\partial \nu}}_{=\langle \nabla u, \nu \rangle}$

$\partial\Omega(\beta)$ is the level set of $\beta \Rightarrow \nu \parallel \nabla u$

$u \nearrow$ along inward $\Rightarrow \nu = -\nabla u / |\nabla u|$

$$\Rightarrow \int_{\Omega(\beta)} \Delta u = -\int_{\partial\Omega(\beta)} |\nabla u|$$

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Again. $\int_{\partial\Omega(\beta)} |\nabla u| \cdot \int_{\partial\Omega(\beta)} |\nabla u|^{-1} \geq (\mathrm{Vol}(\partial\Omega(\beta)))^2$

$$\int_{\Omega(\beta)} \Delta u \leq - (\mathrm{Vol}(\partial\Omega(\beta)))^2 \Big/ \int_{\partial\Omega(\beta)} |\nabla u|^{-1}$$

$$= (\mathrm{Vol}(\partial\Omega(\beta)))^2 \Big/ \mathrm{Vol}(M)\, b'(\bar{u}(\beta))$$

$$= \left(\frac{\mathrm{Vol}(\partial\Omega(\beta))}{\mathrm{Vol}(M)}\right)^2 \frac{\mathrm{Vol}(M)}{b'(\bar{u}(\beta))}$$

$\geq H(\beta)$ $\quad \leq 0$

$$\leq H(\beta)\, F''(\beta)$$

<u>Upshot</u> $\int_{\Omega(\beta)} \Delta u \leq H(\beta)\, F''(\beta)$

(For $\hat{u}$ on (M^*, g^*) which is radially symmetric

$$\int_{\hat{\Omega}(\beta)} \Delta^* \hat{u} = H(\beta)\, \hat{F}''(\beta) \Big)$$

<u>step 2</u> For $u(t,x) : \mathbb{R}_{\geq 0} \times M \to \mathbb{R}_{>0}$

<u>claim</u> $\int_{\Omega(\beta)} \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} F(t,\beta)$

$$\left(\text{reason: } \frac{\partial}{\partial t} \int_{\Omega(\beta)} u(t,\beta) \right)$$

$\Omega(\beta) \leftarrow$ t-dependent

but $Vol(\Omega(\beta)) = \beta \, Vol(M)$

If $\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$

$$\Rightarrow \underbrace{\int_{\Omega(\beta)} \frac{\partial}{\partial t} u}_{\substack{\| \\ \frac{\partial}{\partial t} F(t,\beta)}} - \underbrace{\int_{\Omega(\beta)} \Delta u}_{\substack{\vee \\ H(\beta) \frac{\partial^2}{\partial \beta^2} F(t,\beta)}} = 0$$

<u>step 3</u> Do comparison for the above equation on $(t,\beta) \in \mathbb{R}_{\geq 0} \times [0,1]$

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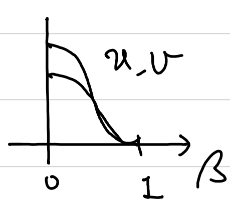
<u>rmk</u> some techniques in the argument



i) $u, v : [0,1] \to \mathbb{R}_{\geq 0}$

$1 \mapsto 0$

monotone decreasing

If $\int_0^\beta u(s)\,ds \geq \int_0^\beta v(s)\,ds \quad \forall \beta$

2nd MVT for integration

$$\int_0^1 (u^2 - v^2) = \int_0^1 (u+v)(u-v)$$

$$= (u(0)+v(0)) \int_0^\beta (u(s)-v(s)) \geq 0$$

$$+ (\cancel{u(1)+v(1)})^{\to 0} \int_\beta^1 (u(s)-v(s))$$

ii) $\displaystyle\int_{y\in M} k^2(t,x,y)\, d\mu_y$

$$k(t,x,y) = \sum_{i\geq 0} e^{-\lambda_i^2 t} \varphi_i(x)\, \varphi_i(y)$$

$$= \int_y \sum_{i,j\geq 0} e^{-(\lambda_i^2+\lambda_j^2)t} \varphi_i(x)\, \varphi_i(y)\, \varphi_j(y)\, \varphi_j(x)\, d\mu_y$$

$$= \sum_{i\geq 0} e^{-2\lambda_i^2 t} \varphi_i(x)\, \varphi_i(x) = k(2t, x, x)$$