§I eigenvalue of a domain

$\Omega \subset \mathbb{R}^n$. smooth, bounded domain
(Ω: open. $\partial\Omega$: smooth)

Let $\lambda_1(\Omega)$ be the 1st Dirichlet eigenvalue of Δ

$$= \inf \left\{ \int_\Omega |\nabla u|^2 \Big/ \int_\Omega |u|^2 : \begin{array}{l} u|_{\partial\Omega} = 0 \\ u \not\equiv 0 \end{array} \right\}$$

- Similarly. L^2_0 has eigenbasis
  (f_i, λ_i) $i = 1, 2, \ldots$
  $0 < \lambda_1 < \lambda_2 \leq \cdots$
  $\Delta f_i = -\lambda_i f_i$. $f_i|_{\partial\Omega} = 0$
  $f_1 > 0$ on Ω
- The only constant function is zero
  $\Rightarrow$ 0 is not an eigenvalue

<u>theorem</u> For any Ω as above. let Ω^* be the open ball in $\mathbb{R}^n$ with
$\mathrm{Vol}(\Omega^*) = \mathrm{Vol}(\Omega)$

Then. $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ Faber – Krahn inequality

sketch of pf: For brevity, write f for f_1

$\forall t > 0$. let $\Omega_t = \{ x \in \Omega : f(x) > t \}$

Let Ω_t^* be the ball with $\mathrm{Vol}(\Omega_t^*) = \mathrm{Vol}(\Omega_t)$

Note that $\Omega^*_{t_1} \subset \Omega^*_{t_2}$ if $t_1 > t_2$
and $\Omega^*_0 = \Omega^*$

Let $f^* : \Omega^* \to \mathbb{R}^+$ be the radial symmetric function with
$$f^*(x) = t \quad \text{for} \quad x \in \partial \Omega^*_t$$
It follows that f is non-increasing in the radius (= distance to the origin)

Intuitively, f^* is the "symmetrization" of f

<u>Recall</u> coarea formula
$$\int_{f \leq a} u \, |df| = \int_0^a \left(\int_{f^{-1}(t)} u \right) dt$$
or
$$\int_{f \leq a} u = \int_0^a \left(\int_{f^{-1}(t)} u \, |df|^{-1} \right) dt \quad —(*)$$

Now, consider $a = \sup_\Omega f$, $u = |df|^2$
$$\Rightarrow \int_\Omega |df|^2 = \int_0^a \left(\int_{f^{-1}(t)} |df| \right) dt$$

Also, consider $(*)$ with $u = 1$, and take $\frac{d}{da}$ $\Rightarrow \frac{d}{dt} \mathrm{Vol}(\Omega_t) = \int_{f^{-1}(t)} |df|^{-1}$

$$\int_{f^{-1}(t)} |df|^{-1} \int_{f^{-1}(t)} |df| \geq \int_{f^{-1}(t)} 1 = \underset{\partial\Omega_t}{\underset{\parallel}{Vol(f^{-1}(t))}}$$

Similarly. $\displaystyle\int_{\Omega} |df^*|^2 = \int_0^a \left(\int_{(f^*)^{-1}(t)} |df^*| \right) dt$

Since $|df^*| = \text{const}$ on $(f^*)^{-1}(t)$

$$\int_{(f^*)^{-1}(t)} |df^*|^{-1} \int_{(f^*)^{-1}(t)} |df^*| = \int_{(f^*)^{-1}(t)} 1 = Vol(\partial\Omega_t^*)$$

<u>recall</u> classical isoperimetric inequality

$$\frac{Vol(\partial\Omega_t)^n}{Vol(\Omega_t)^{n-1}} \geq \frac{Vol(\partial\Omega_t^*)^n}{Vol(\Omega_t^*)^{n-1}}$$

and $Vol(\Omega_t) = Vol(\Omega_t^*)$

by construction

$$\Rightarrow \int_{f^{-1}(t)} |df| \geq \frac{Vol(\partial\Omega_t)}{\int_{f^{-1}(t)} |df|^{-1}} \geq \frac{Vol(\partial\Omega_t^*)}{\int_{f^{-1}(t)} |df|^{-1}}$$

$$= \underbrace{\left(\int_{(f^*)^{-1}(t)} |df^*|^{-1} \Big/ \int_{f^{-1}(t)} |df|^{-1} \right)}_{\frac{d}{dt} Vol(\Omega_t^*) \big/ \frac{d}{dt} Vol(\Omega_t) \, = \, 1} \int_{(f^*)^{-1}(t)} |df^*|$$

Hence $\int_\Omega |df|^2 \geq \int_{\Omega^*} |df^*|^2$

$$\int_\Omega f^2 = \int_0^a \Big(\underbrace{\int_{f^{-1}(t)} f^2 |df|^{-1}}_{\substack{\| \\ t^2 \int_{f^{-1}(t)} |df|^{-1} = t^2 \int_{(f^*)^{-1}(t)} |df^*|^{-1}}}\Big)\, dt = \int_{\Omega^*} (f^*)^2$$

as before

$$\Rightarrow \quad \lambda_1(\Omega) = \frac{\int_\Omega |df|^2}{\int_\Omega f^2} \geq \frac{\int_{\Omega^*} |df^*|^2}{\int_{\Omega^*} (f^*)^2}$$

$$\geq \inf \cdots = \lambda_1(\Omega^*)$$

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§ II On a Riemannian manifold

1° For $\Omega \subset M$, we can still consider the least area of $\partial\Omega$

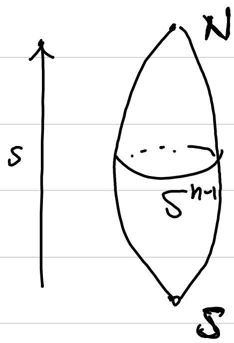
defn The Isoperimetric function of (M,g) for $\beta \in [0,1]$ is

$$h(\beta) = h(M,g;\beta) = \inf \left\{ \frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(M)} : \Omega \subset M,\ \mathrm{Vol}(\Omega) = \beta\, \mathrm{Vol}(M) \right\}$$

An Isoperimetric estimator of (M,g) is a function $H : [0,1] \to \mathbb{R}_{\geq 0}$ such that $h(\beta) \geq H(\beta) \quad \forall \beta \in [0,1]$

prop.
- $h(\beta) = h(1-\beta)$
- $h(\beta) \sim c_n \, Vol(M)^{-\frac{1}{n}} \beta^{\frac{n-1}{n}}$ for $\beta \sim 0$
- $h(\beta)$ is continuous, has left and righ derivative $\forall \beta$, and is differentiable except on a denumerable set

2° From now on, let us assume (M,g) admits an Isoperimetric estimator $H(\beta)$ and see how to use it



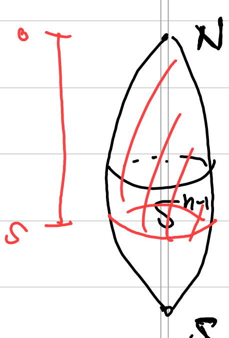
$$M^* = S^{n-1} \times (0,L) \cup \{N, S\}$$

$$g^* = ds^2 + a^2(s)\, g_0 \leftarrow \text{standard metric on } S^{n-1} \text{ of radius } 1$$

$$a(0) = 0 = a(L)$$

$$a(s) > 0 \quad \forall s \in (0,L)$$

$$(\text{if } a'(0) = 1,\ a'(L) = -1 \Rightarrow \text{smooth})$$



Let $V^* = Vol(M^*, g^*)$

$$A(s) = Vol(\text{ball of radius } s \text{ at } N) / V^* \quad \text{ball of radius } s \text{ at } N = B_s(N)$$

$$= \frac{Vol(S^{n-1})}{V^*} \int_0^s a^{n-1}(\rho)\, d\rho \in [0,1]$$

How to choose $a(s)$?

It is natural to require that

$$A'(s) = \frac{Vol(\partial B_s(N))}{V^*} = H(A(s)) \quad \forall s$$

$$\Rightarrow \quad \frac{1}{H(A(s))} A'(s) = 1$$

$$\Rightarrow \quad \int_0^{A(s)} \frac{d\beta}{H(\beta)} = s \quad : \text{ this determines } A(s)$$

and thus $\displaystyle\int_0^1 \frac{d\beta}{H(\beta)} = L$

3° example (S^2, g_0)

$$g = dr^2 + \sin^2 r \, d\theta^2$$

$$\mathrm{Vol}(B_r(N)) = \int_0^r \int_0^{2\pi} \sin s \, ds \, d\theta$$

$$= 2\pi(1 - \cos r)$$

$$\mathrm{Vol}(\partial B_r(N)) = 2\pi \sin r$$

$$\mathrm{Vol}(S^2) = 4\pi$$

$$\Rightarrow \text{ If } \beta = \frac{2\pi(1-\cos r)}{4\pi} = \frac{1-\cos r}{2}$$

$$h(\beta) = \frac{2\pi \sin r}{4\pi} = \frac{1}{2} \sin r = \sqrt{\beta(1-\beta)}$$

$$\cos r = 1 - 2\beta \qquad \sin r = \sqrt{1-(1-2\beta)^2} = \sqrt{2\beta(2-2\beta)}$$

exercise

Now, suppose that (M^2, g) has isoperimetric estimator $H(\beta) = \sqrt{\beta(1-\beta)}$

$$\Rightarrow \int_0^{A(s)} \frac{d\beta}{\sqrt{\beta(1-\beta)}} = s \quad \Rightarrow \quad A(s) = \sin^2 \frac{s}{2}, \quad L = \pi$$

$$A(s) = \frac{2\pi}{V^*} \int_0^s a(\rho)\, d\rho$$

$$\Rightarrow a(s) = \frac{V^*}{4\pi} \sin s$$

$$g^* = ds^2 + \left(\frac{V^*}{4\pi}\sin s\right)^2 d\theta^2$$

Note that V^* is a formal parameter

If we take $V^* = 4\pi$

$\Rightarrow (M^*, g^*) = (S^2, g_0)$

4° Cheeger's inequality

Cheeger's isoperimetric constant $\rightarrow$ $C_0 = \inf \left\{ \mathrm{Vol}(\partial\Omega) / \mathrm{Vol}(\Omega) : \Omega \subset M, \ \mathrm{Vol}(\Omega) \le \frac{1}{2}\mathrm{Vol}(M) \right\}$

For $\beta \in [0, \frac{1}{2}]$

$\mathrm{Vol}(\Omega) = \beta \, \mathrm{Vol}(M)$

$$\Rightarrow \underbrace{\mathrm{Vol}(\partial\Omega) / \mathrm{Vol}(\Omega)}_{\ge\, h_c} = \frac{1}{\beta} \underbrace{\mathrm{Vol}(\partial\Omega) / \mathrm{Vol}(M)}_{\ge\, h(\beta)}$$

$\Rightarrow h_\beta \ge \beta C_0$

For $\beta \in [0, 1]$, $h_\beta \ge C_0 \min\{\beta, 1-\beta\}$

thm (Cheeger) $\quad \lambda_1(M, g) \ge \frac{C_0^2}{4}$

sketch (0) $\lambda_0(M, g) = 0 \qquad \varphi_0 = 1/\sqrt{\mathrm{Vol}(M)}$

$\lambda_1 > 0. \qquad \int_M \varphi_1 = 0$

$\Gamma = \varphi_1^{-1}(0)$

Ω = one-connected component of $M \setminus \Gamma$ with $\mathrm{Vol}(\Omega) < \frac{1}{2}\mathrm{Vol}(M)$

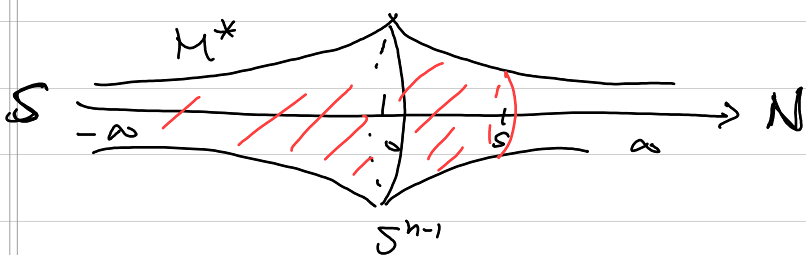
We may assume $\varphi_1 > 0$ on Ω

(i) Consider $H(\beta) = c_0 \min\{\beta, 1-\beta\}$

Note that $\int_0^1 \frac{1}{H(\beta)}\, d\beta = 2\int_0^{1/2} \frac{1}{\beta}\, d\beta = \infty$

Consider $M^* = S^{n-1} \times (-\infty, \infty)$

$g^* = ds^2 + a^2(s)\, g_0$



$$(\beta =)\, A(s) = \frac{Vol(S^{n-1})}{V^*} \int_{-\infty}^{s} a^{n-1}(t)\, dt$$

$$= \frac{1}{2} + \frac{Vol(S^{n-1})}{V^*} \int_0^s a^{n-1}(t)\, dt$$

$A(0) = \frac{1}{2}$

By the symmetry of H, $H(\beta) = H(1-\beta)$.

$a(s) = a(-s)$

$A(s) + A(-s) = 1$

Require $\frac{Vol(\partial B_s(N))}{V^*} = H(\beta)$

$$\underset{A'(s)}{\parallel} \qquad \underset{H(A(s))}{\parallel}$$

$\Rightarrow \quad s = \int_{\frac{1}{2}}^{A(s)} \frac{1}{c_0(1-\beta)}\, d\beta \quad$ for $s \geq 0$

$\Rightarrow \quad c_0 s = -\log(2(1-A(s)))$

$\Rightarrow \quad A(s) = 1 - \frac{1}{2}\exp(-C_0 s)$

$\left(\left(\frac{1}{2} + \frac{Vol(S^{n-1})}{V^*}\int_0^s a^{n-1}(t)\,dt\right.\right.$

$\Rightarrow \quad \frac{Vol(S^{n-1})}{V^*} a^{n-1}(s) = \frac{C_0}{2}\exp(-C_0 s)$

Choose $V^* = Vol(M) = V$

(ii) $\Omega_t = \{ x \in \Omega : \varphi_1(x) > t \}$ $\Rightarrow \Omega = \Omega_0$

$\Omega_t^* = S^{n-1} \times (S(t), \infty)$ $\quad \Omega^* := \Omega_0^*$

with $Vol(\Omega_t^*) = Vol(\Omega_t)$

Let $\psi : S^{n-1} \times (S(0), \infty) \to \mathbb{R}_{>0}$

defined by $\psi(\theta, S(t)) = t$

With the same argument as that for Faber-Krahn inequality,

$$\lambda_1(M,g) = \frac{\int_\Omega |d\varphi_1|^2}{\int_\Omega |\varphi_1|^2} \geq \frac{\int_\Omega |d\psi|^2}{\int_\Omega |\psi|^2}$$

$$= \frac{\int_{r(0)}^\infty (\psi'(s))^2 \exp(-C_0 s)\,ds}{\int_{r(0)}^\infty (\psi(s))^2 \exp(-C_0 s)\,ds}$$

1D problem

$\psi(r(0)) = 0$

can shown to be $\geq C_0^2/4$