§I Euclidean space

$$\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i}\right)^2 \quad \text{on} \quad \mathbb{R}^n$$

recall $\quad k(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$

$$\in C^\infty\left((0,\infty) \times \left(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x=y\}\right)\right)$$

It has the following properties

(i) $k(t,x,y) = k(t,y,x)$

(ii) $\left(\frac{\partial}{\partial t} - \Delta_x\right) k = 0$

(iii) $\int_{y \in \mathbb{R}^n} k(t,x,y) f(y)\, dy \xrightarrow{t \to 0^+} f(x)$

$\tilde{f}(t,x) :=$

(iv) $\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{f} = 0 \\ \tilde{f}(0,x) = f(x) \end{cases}$

instantaneously smoothing

Namely, k is the fundamental solution to the heat equation subject to the initial condition $(t=0)$

§II formal expression

Now, work on (M,g)

$\Delta \varphi_i = -\lambda_i \varphi_i \qquad \lambda_i \geq 0 \qquad \{\varphi_i\}$: basis for $L^2(M)$

$f = \sum_{i \geq 0} a_i \varphi_i$

Solve $\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{f} = 0 \\ \tilde{f}(0,x) = f(x) \end{cases}$

Similarly, write $\tilde{f}(t,x) = \sum_{i \geq 0} c_i(t)\, \varphi_i(x)$

$$c_i(0) = a_i$$

$$\sum_{i \geq 0} (c_i' + \lambda_i c_i)\, \varphi_i = 0$$

$$\Rightarrow \quad c_i(t) = a_i e^{-\lambda_i t}$$

Therefore,

$$\tilde{f}(t,x) = \sum_{i \geq 0} \underline{a_i}\, e^{-\lambda_i t} \varphi_i(x)$$

$$= \sum_{i \geq 0} \underline{(f, \varphi_i)}\, e^{-\lambda_i t} \varphi_i(x)$$

$$= \int_{y \in M} f(y) \left(\sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \right) d\mu$$

$$\Rightarrow \quad e(t,x,y) = \sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(x)\, \varphi_i(y)$$

symmetric in x & y.

is the heat kernel

rmk One can show that

$$e(t,x,y) \in C^\infty\left((0,\infty) \times M \times M\right)$$

based on the discussion last time

§ III. parametrix construction

0° $\Delta(f_1 f_2) = (\Delta f_1) f_2 + 2\langle df_1, df_2 \rangle + f_1 (\Delta f_2)$

pointwise formula. direct computation

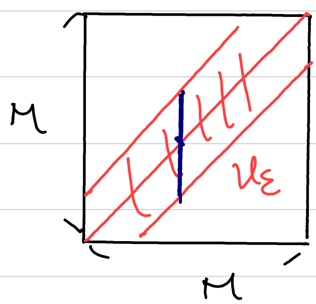
1° Naturally, consider

$$G(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathrm{dist}^2(x,y)}{4t}\right)$$

$$\in C^\infty(\mathbb{R}^+ \times \mathcal{U}_\varepsilon)$$

Fix $0 < \varepsilon <$ injectivity radius of M

$\mathcal{U}_\varepsilon = \{(x,y) \in M \times M : \mathrm{dist}(x,y) < \varepsilon\}$



In general,

$$\left(\frac{\partial}{\partial t} - \Delta_y\right) G \neq 0$$

2° For any $x \in M$, use the geodesic polar coordinate at x

$(s^1 \cdots s^n) \in \mathbb{R}^n \cong T_x M$ $\quad T_xM \ni 0$

$\longmapsto \exp_x(s^i e_i) \in M$ $\quad$ the point y $\quad x$

Then, use the "polar" coordinate for $\mathbb{R}^n$

$$[0,\infty) \times S^{n-1} \longrightarrow \mathbb{R}^n$$

$$(r, \theta_2, \cdots, \theta_n)$$

For $n=2$, $(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$

By Gauss lemma,

$$g = dr^2 + r^2 \sum_{2 \le i,j \le n} \tilde{g}_{ij}(r,\theta)\, d\theta_i\, d\theta_j$$

- no $dr\, d\theta_i$ terms
- $\lim_{r\to 0} \sum_{2\le i,j\le n} \tilde{g}_{ij}(r,\theta)\, d\theta_i\, d\theta_j \longrightarrow g_{std}$ $\quad S^{n-1}$

- The series expansion of $\tilde{g}$ is given by curvature & its covariant derivatives.
- For $n=2$

$$dr^2 + r^2\left(1 - \frac{K}{6}r^2 + \cdots\right)^2 d\theta^2$$

Gaussian curvature at x

$$\Rightarrow \det g = r^{2n-2} \det \tilde{g}_{ij} =: D^2$$

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{1\le i,j\le n} \left(\frac{\partial}{\partial \xi^i} \sqrt{\det g}\, g^{ij} \frac{\partial}{\partial \xi^j} \right)$$

For $f = f(r)$, only $j=1 \Rightarrow i=1$

$$\Rightarrow \Delta f = \frac{1}{r^{n-1}D} \frac{\partial}{\partial r}\left(r^{n-1} D \frac{\partial f}{\partial r} \right)$$

$$= \frac{\partial^2 f}{\partial r^2} + \left(\frac{n-1}{r} + \frac{D'}{D} \right) \frac{\partial f}{\partial r}$$

$\partial_r D$

(Also, $df = \frac{\partial f}{\partial r} dr$)

3° (educated) guess :

Fix $k \in \mathbb{N}$. consider

$$S = (4\pi t)^{-\frac{n}{2}} \exp\left(- \frac{\operatorname{dist}^2(x,y)}{4t}\right) \cdot \left(u_0(x,y) + u_1(x,y)\, t + \cdots + u_k(x,y)\, t^k \right)$$

$$= \underbrace{(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{4t}\right)}_{G} \sum_{j\ge 0} u_j(r,\theta)\, t^j$$

Note that $\frac{\partial G}{\partial t} = \left(-\frac{n}{2t} + \frac{r^2}{4t^2}\right) G$

$$\frac{\partial G}{\partial r} = -\frac{r}{2t} G$$

$$\Delta G = \left(-\frac{1}{2t} + \frac{r^2}{4t^2} - \frac{n-1}{2t} - \frac{r}{2t}\frac{D'}{D}\right) G$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right) G = \frac{r}{2t}\frac{D'}{D} G$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(G \cdot (u_0 + u_1 t + \cdots + u_k t^k)\right)$$

$$= \frac{r}{2t}\frac{D'}{D} G \cdot (u_0 + u_1 t + \cdots + u_k t^k)$$

$$+ G(u_1 + 2u_2 t + \cdots + k u_k t^{k-1})$$

$$- G(\Delta u_0 + \Delta u_1 t + \cdots + \Delta u_k t^k)$$

$$\underbrace{- 2\langle dG, du_0 + du_1 t + \cdots + du_k t^k\rangle}_{+ \frac{r}{t} G\left(\frac{\partial u_0}{\partial r} + \frac{\partial u_1}{\partial r} t + \cdots + \frac{\partial u_k}{\partial r} t^k\right)}$$

$$dG = -\frac{r}{2t} G\, dr$$

$t^{-1} G$: $\quad r\frac{\partial u_0}{\partial r} + \frac{r}{2}\frac{D'}{D} u_0 = 0$

G: $\quad r\frac{\partial u_1}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + 1\right) u_1 - \Delta u_0 = 0$

$t^{i-1} G$: $\quad r\frac{\partial u_i}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + i\right) u_i - \Delta u_{i-1} = 0$

$t^k G$: $\quad -\Delta u_k$ (left-over)

$u_0 = D^{-\frac{1}{2}}$ $\quad$ note that $u_0(0) = 1$

$$r\frac{\partial u_i}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + i\right) u_i - \Delta u_{i-1} = 0$$

The integral facter is $r^i D^{\frac{1}{2}}$

$$v_i = r^i D^{\frac{1}{2}} u_i$$

$$\Rightarrow \quad \frac{\partial v_i}{\partial r} = - r^{i-1} D^{\frac{1}{2}} \Delta u_{i-1}$$

$$\Rightarrow \quad u_i(r,\theta) = - r^{-i} D^{-\frac{1}{2}}(r,\theta) \cdot \int_0^r \rho^{i-1} D^{\frac{1}{2}}(\rho,\theta) (\Delta u_{i-1})(\rho,\theta)\, d\rho$$

Finally, $\left(\frac{\partial}{\partial t} - \Delta_y\right) S_k$

$$= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{dist^2(x,y)}{4t}\right) t^k \Delta_y u_k$$

Choose a cut $\chi(r) = \begin{cases} 1 & r \le \varepsilon/2 \\ 0 & r \ge \varepsilon \end{cases}$

$$\Rightarrow \quad \chi(dist(x,y)) S_k \in C^\infty((0,\infty) \times M \times M)$$

<u>Lem</u> (i) $\left(\frac{\partial}{\partial t} - \Delta\right)(\chi S_k) \in C^\ell([0,\infty) \times M \times M)$

if $\ell < k - \frac{n}{2}$

(ii) $\lim_{t \to 0} \int_{y \in M} (\chi S_k)(t,x,y) f(y)\, d\mu = f(x)$

pf: direct computation and calculus

key for (ii) $\quad u_0(x,x) = 1$ $\quad$ ※

§IV. True heat kernel ?

$A : V \circlearrowleft$ $\quad$ V: some vector space

Denote by e^{tA} (if exist !) the one-parameter family of operator from V to V such that

$$\begin{cases} (\partial_t - A)(e^{tA} v) = 0 \\ e^{tA} v \longrightarrow v \quad \text{as} \quad t \to 0 \end{cases}$$

<u>goal</u> relate $e^{t(A+B)}$ to e^{tA} for $A, B : V \circlearrowleft$

Consider $e^{t(A+B)} e^{-tA}$

$$\frac{d}{dt} e^{t(A+B)} e^{-tA} = e^{t(A+B)} (A+B) e^{-tA}$$

commutes

$$- e^{t(A+B)} A e^{-tA}$$

$$= e^{t(A+B)} B e^{-tA}$$

$$\Rightarrow e^{t(A+B)} e^{-tA} - I$$

$$= \int_0^t e^{s(A+B)} B e^{-sA} \, ds$$

$$\Rightarrow e^{t(A+B)} = e^{tA} + \underbrace{\int_0^t e^{s(A+B)} B e^{(t-s)A} \, ds}_{= \, e^{t(A+B)} * (Be^{tA})}$$

Duhamel's principle

convolution

$$
\begin{aligned}
e^{*(A+B)} &= e^{*A} + e^{*(A+B)} * (Be^{*A}) \\
&= e^{*A} + \left[e^{*A} + e^{*(A+B)} * (Be^{*A})\right] * (Be^{*A}) \\
&= e^{*A} + e^{*A} * (Be^{*A}) + e^{*(A+B)} * (Be^{*A})^{*2} \\
&\vdots \\
&= e^{*A} + \sum_{j=1}^{n-1} e^{*A} * (Be^{*A})^{*j} \\
&\qquad + e^{*(A+B)} * (Be^{*A})^{*n}
\end{aligned}
$$

If $e^{*(A+B)} * (Be^{*A})^{*n} \longrightarrow 0$

then $e^{*(A+B)} = e^{*A} + \sum_{j \geq 1} e^{*A} * (Be^{*A})^{*j}$

Now, $e^{*\Delta}$: the heat kernel for Δ

$\int_{y \in M} \eta S_k(t, x, y) f(y) \, d\mu$: $C^\infty(M) \circlearrowleft$

pretend it solves $\frac{\partial}{\partial t} - A$

for some operator A (!?)

Namely e^{*A} is given by the integral operator with kernel ηS_k

$\Rightarrow$ $\Delta = A + B$. $\quad e^{*(A+B)}$: goal

e^{*A} : known

$\Rightarrow$ $Be^{*A} = (\Delta - A) e^{*A} = (\Delta - \frac{\partial}{\partial t}) e^{*A}$

$\Rightarrow$ $e^{*\Delta} = e^{*A} + \sum_{j \geq 1} e^{*A} * \left((\Delta - \frac{\partial}{\partial t}) e^{*A}\right)^{*j}$

§IV true heat kernel

last time: $e^{t(A+B)} \text{ "=" } e^{tA} + \sum_{j\geq 1} e^{tA} * (Be^{tA})^{*j}$

goal: $e^{t\Delta}$: fundamental solution to

$$\frac{\partial}{\partial t} - \Delta = 0$$

$$\int_{y \in M} \eta S_k(t,x,y) f(y)\, d\mu \quad : \quad C^\infty(M) \circlearrowleft$$

pretend it solves $\frac{\partial}{\partial t} - A$

for some operator A (!?)

Namely e^{tA} is given by the integral operator with kernel ηS_k

(recall: η = cut-off, supported near the diagonal in $M \times M$

$$S_k(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathrm{dist}^2(x,y)}{4t}\right) \sum_{j=0}^{k} u_j(x,y)\, t^k$$
)

$\Rightarrow$ $\Delta = A + B$. $\quad e^{t(A+B)}$: goal

e^{tA} : known

$\Rightarrow$ $Be^{tA} = (\Delta - A)e^{tA} = \left(\Delta - \frac{\partial}{\partial t}\right)e^{tA}$

$\Rightarrow$ $e^{t\Delta} \text{ "=" } e^{tA} + \sum_{j\geq 1} e^{tA} * \left(\left(\Delta - \frac{\partial}{\partial t}\right)e^{tA}\right)^{*j}$

$(\Delta - \frac{\partial}{\partial t}) e^{tA} : C^\infty(M) \circlearrowleft$

$$f(x) \longmapsto (e^{tA} \cdot f)(t,x) = \int \eta\, S_k(t,x,y) f(y)\, dy$$

$$\longmapsto \left((\Delta - \tfrac{\partial}{\partial t}) e^{tA} \cdot f \right)(t,x)$$

$$= \int (\Delta_x - \tfrac{\partial}{\partial t})(\eta\, S_k(t,x,y)) f(y)\, dy$$

For $P_t : C^\infty(M) \circlearrowleft$ and $Q_t : C^\infty(M) \circlearrowleft$

given by $\int p(t,x,y) f(y)\, dy$

and $\int q(t,x,y) f(y)\, dy$

$\Rightarrow P_t * Q_t$ is given by

$$\int_0^t \int_{y \in M} \int_{z \in M} p(s,x,z)\, q(t-s,z,y)\, f(y)\, dy\, dz\, ds$$

Namely, its a (t-family of) operators with integral kernel

$$\int_0^t \int_{z \in M} p(s,x,z)\, q(t-s,z,y)\, dz$$

Instead of $\Delta_x - \frac{\partial}{\partial t}$ it is easier to work with $\Delta_y - \frac{\partial}{\partial t}$ in the construction (since we do so for constructing S_k)

$$\left(\frac{\partial}{\partial t} - \Delta_y\right)\left(P * (\eta S_k)\right)$$

$$= \left(\frac{\partial}{\partial t} - \Delta_y\right)\left(\int_0^t ds \int_{z\in M} P(s,x,z)(\eta S_k)(t-s,z,y)\,dz\right)$$

$$= \lim_{s\to t} \int_{z\in M} P(s,x,z)(\eta S_k)(t-s,z,y)\,dz$$

$$+ \int_0^t ds \int_{z\in M} P(s,x,z)\left(\frac{\partial}{\partial t} - \Delta_y\right)(\eta S_k)(t-s,z,y)\,dz$$

$$= P + P * \underbrace{\left(\frac{\partial}{\partial t} - \Delta_y\right)(\eta S_k)}_{K_k}$$

Let $K_k = \left(\Delta_y - \frac{\partial}{\partial t}\right)(\eta S_k)$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} - \Delta_y\right)(\eta S_k) = K_k$$

$$\left(\frac{\partial}{\partial t} - \Delta_y\right)\left(K_k * (\eta S_k)\right) = K_k + K_k * K_k$$

$$\left(\frac{\partial}{\partial t} - \Delta_y\right)\left(K_k^{*2} * (\eta S_k)\right) = K_k^{*2} + K_k^{*3}$$

$$\vdots$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} - \Delta_y\right)\left(\eta S_k - \left(K_k - K_k^{*2} \pm \cdots\right) * (\eta S_k)\right) = 0$$

<u>lem</u> $Q_k = \sum_{j\geq 1} (-1)^{j-1} K_k^{*j}$ exists and is in $C^l([0,\infty)\times M \times M)$ if $k - \frac{n}{2} > l$. Moreover $\forall\, T > 0$ $\exists\, C_T$ such that $|Q_k| \leq C t^{k-\frac{n}{2}}$ $\forall\, t \in [0, T]$

sketch: $(\Delta_y - \frac{\partial}{\partial t}) S_k$

$$= -(4\pi)^{-\frac{n}{2}} \exp\left(-\frac{\mathrm{dist}^2(x,y)}{4t}\right) t^{k-\frac{n}{2}} \Delta u_y$$

near diagonal

direction computation

$$\Rightarrow (K_k)^{*j} \lesssim \frac{c^j \, t^{k-\frac{n}{2}+j-1}}{j!} \quad \text{as } j \gg 1$$

<u>upshot</u> $\eta S_k - Q_k * (\eta S_k)$ satisfies

$$\frac{\partial}{\partial t} - \Delta_y = 0$$

and one can show it converges to the delta distribution ($\because u_0(x,x) \equiv 1$) as $t \to 0^+$