On (M, g), denote by Δ the Laplace-Beltrami operator on real-valued functions

In coordinate, $\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j}\right)$

$= -(\delta d + \cancel{d\delta}^{\,0})$

§ I. uniqueness of the solution to heat

lem Given $f \in C^\infty(M)$, let

$\tilde{f} \in C^0([0, T) \times M) \times C^\infty((0, T) \times M)$

such that $\begin{cases} (\frac{\partial}{\partial t} - \Delta)\tilde{f} = 0 \\ \tilde{f}(0, x) = f(x) \end{cases}$

Then, $\tilde{f}$ is unique

Pf: It suffices to show that $\tilde{f}$ must vanish if f is zero

$$\frac{d}{dt} \int_M |\tilde{f}|^2 d\mu = \int 2\tilde{f}\, \partial_t \tilde{f}\, d\mu$$

$$= \int 2\tilde{f}\, \Delta\tilde{f}\, d\mu$$

$$= -2\int |\nabla \tilde{f}|\, d\mu$$

$\Rightarrow \int |\tilde{f}|^2 d\mu$ is non-increasing in t

Since f is zero at $t=0$,

$\int |\tilde{f}|^2 d\mu \equiv 0 \quad \forall t \Rightarrow \tilde{f} \equiv 0$ ※

§II. coarse estimate on eigenvalues

1° <u>recall</u> from exercise in ch. 6 of Warner

$\exists (\lambda_i, \varphi_i) \in \mathbb{R}_{\geq 0} \times C^\infty(M)$ for $i = 0, 1, \dots$

such that
- $\Delta \varphi_i = -\lambda_i \varphi_i$
- $0 = \lambda_0 < \lambda_1 \leq \cdots \longrightarrow \infty$ (no finite accumulation value)
- $\{\varphi_i\}_{i \geq 0}$ is a basis for $L^2(M)$

$\varphi_0 = (\mathrm{Vol}(M))^{1/2}$

2° The growth rate of λ_i in i?

Fix $k \in \mathbb{N}$

Let $E_k = \mathrm{span}\{\varphi_0, \dots \varphi_k\}$

<u>goal</u> study $\max |f|$ for $f \in E_k$

i) $H_s = W^{2,s}$

For $s \in \mathbb{N}$. $\|f\|_{H_s}^2 \approx \int |f|^2 + \dots + |\nabla^{(k)} f|^2$

Elliptic estimate. $\|f\|_{H_2} \lesssim \|(1-\Delta) f\|_{L^2}$

$\Rightarrow \|f\|_{H_{2m}} \lesssim \|(1-\Delta)^m f\|_{L^2}$

ii) By Sobolev. fix $\delta > 0$

$H_{\delta + n/2} \hookrightarrow C^0$

For $f \in E_k$

$$\|f\|_{L^\infty} \lesssim \|f\|_H \lesssim (1+\lambda_k)^{\frac{1}{2}(\delta + \frac{n}{2})} \|f\|_{L^2}$$

iii) For any $c_0, \dots, c_k \in \mathbb{R}$

Consider $S : L^2(M) \to \mathbb{R}$

$$f \mapsto (f, \sum_{0 \le i \le k} c_i \varphi_i)$$

$$\Rightarrow |S(f)| \le \|f\|_{L^2} \sqrt{\sum_{0 \le i \le k} c_i^2}$$

In fact, the operator norm of S is exactly $\sqrt{\sum_{0 \le i \le k} c_i^2}$

iv) For any $y \in M$, consider S_y for $c_i = \varphi_i(y)$

$$\Rightarrow \|S_y\|_{op}^2 = \sum_{0 \le i \le k} \varphi_i^2(y)$$

But for $f \in L^2(M) \Rightarrow f(x) = \sum_{i \ge 0} a_i \varphi_i(x)$

$$\downarrow$$

$\pi_k(f) \in E_k$ given by $\sum_{0 \le i \le k} a_i \varphi_i(x)$

$$S_y(f) = \sum_{0 \le i \le k} a_i c_i = \sum_{0 \le i \le k} a_i \varphi_i(y)$$

$$= (\pi_k(f))(y)$$

$$\Rightarrow |S_y(f)| = |\pi_k(f)(y)|$$

$$\lesssim (1+\lambda_k)^{\frac{1}{2}(\delta + \frac{n}{2})} \|\pi_k(f)\|_{L^2}$$

$$\lesssim (1+\lambda_k)^{\frac{1}{2}(\delta + \frac{n}{2})} \|f\|_{L^2}$$

Hence, $\sum_{0 \le i \le k} \varphi_i^2(y) \le C (1+\lambda_k)^{\delta + \frac{n}{2}}$

$\forall k$