# DIFFERENTIAL GEOMETRY II: HOMEWORK 5

DUE MARCH 31

(1) Compute the Euler class of the tangent bundle of $S^4$ by using the standard round metric of radius 1, and evaluate its integral over $S^4$.

(2) Let $E \to M$ be a real vector bundle of rank $k$, and let $\nabla$ be a connection on $E$. Fix a point $O \in M$, and choose a coordinate chart at $O$. By doing so, we may pretend that everything happens on a ball at the origin in $\mathbb{R}^n$. Construct a local trivialization of $E$ as follows.
- Choose a basis $\{e_1, \cdots, e_k\}$ for $E|_0$.
- Parallel transport $\{e_1, \cdots, e_k\}$ along radial line segments emanating from the origin.
- It gives a local frame for $E$. One can show its smoothness by the ODE theory.

With respect to this trivialization, write $\nabla = \mathrm{d} + \sum_{i=1}^n A_i(x)\mathrm{d}x^i$, where each $A_i(x)$ is an $\mathrm{M}(k \times k, \mathbb{R})$-valued function.

(a) What is the $\mathrm{M}(k \times k, \mathbb{R})$-valued function $\sum_{i=1}^n x^i A_i(x)$? Give your reason.

(b) Let
$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \ .$$
Show that
$$A_i(x) = \int_0^1 \sum_{j=1}^n s\, x^j\, F_{ji}(sx)\, \mathrm{d}s$$
for $i = 1, \ldots, n$.

In particular, $A_i(0) = 0$, and $(\mathrm{d}F_{ij})(0) = 0$.

(3) Given a *complex* vector bundle $E \to M$ of rank $k$, one can form a *real* vector bundle $E_0 \to M$ of rank $2k$ by *forgetting the complex structure.*

(a) To be more precise, write the transition of $E$ as the real and imaginary part, $\mathrm{GL}(k, \mathbb{C}) \ni g_{\alpha\beta} = u_{\alpha\beta} + iv_{\alpha\beta}$. Verify that
$$\begin{bmatrix} u_{\alpha\beta} & -v_{\alpha\beta} \\ v_{\alpha\beta} & u_{\alpha\beta} \end{bmatrix} \in \mathrm{GL}(2k, \mathbb{R}) \ ,$$
and satisfies the cocycle condition. It defines the bundle $E_0$.

(b) Consider $\mathbb{R}^{2k}$ with a basis $e_1, \cdots, e_{2k}$. For $\mathbb{R}^{2k} \otimes \mathbb{C} \cong \mathbb{C}^{2k}$, $e_1, \cdots, e_{2k}$ is also a basis (over $\mathbb{C}$). On the other hand, $\{e_j - ie_{j+k}\}_{j=1}^k \cup \{e_j + ie_{j+k}\}_{j=1}^k$ is also a basis for

$\mathbb{C}^{2k}$. Construct two *complex* vector subbundles of $E_0 \otimes \mathbb{C}$ by this choice of basis. Explain their relations to the original complex bundle $E$.

(c) Show that $p_1(E_0) = c_1(E) \wedge c_1(E) - 2c_2(E)$.

(4) The *mean curvature vector* of $\Sigma$ in $(M, \bar{g})$ is the trace of the second fundamental form:

$$\vec{H} = \mathrm{tr}_g(\mathrm{II}) = g^{ij}\, h^{\alpha}_{ij}\, H_\alpha \ .$$

Consider the Clifford torus

$$\Sigma = \{\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) \in \mathbb{R}^4\} \ .$$

(a) Compute the mean curvature vector of $\Sigma \subset \mathbb{R}^4$.

(b) Compute the mean curvature vector of $\Sigma \in S^3$.