DIFFERENTIAL GEOMETRY II: HOMEWORK 4

DUE MARCH 24

(1) For a complex vector bundle $E \to M$, the *total Chern class* is defined to be

$$c(E) = \det\left(\mathbf{I} + \frac{i}{2\pi}F_\nabla\right) = 1 + c_1(E) + c_2(E) + \cdots \in \bigoplus_{j\in\mathbb{N}} \mathrm{H}^{2j}_{\mathrm{dR}}(M) \ .$$

Suppose that E, F are two complex vector bundles over M. Prove that

$$c(E \oplus F) = c(E) \wedge c(F) \ .$$

Namely, $c_j(E\oplus F) = \sum_{\ell=0}^{j} c_\ell(E) \wedge c_{j-\ell}(F) \in \mathrm{H}^{2j}_{\mathrm{dR}}(M)$, where c_0 is set to be 1.

(2) For a complex vector bundle $E \to M$, let E^* be its dual bundle. Prove that $c_j(E^*) = (-1)^j c_j(E)$.

For instance, if $c_1(E) \neq 0 \in \mathrm{H}^2_{\mathrm{dR}}(M)$, E and E^* will not be isomorphic.

(3) On a closed, oriented surface Σ, choose a point p, and a coordinate neighborhood of p, $\mathcal{U}$. Assume that $\mathrm{d}x \wedge \mathrm{d}y$ is the (positive) orientation. Let $\mathcal{V} = \Sigma \backslash \{p\}$

For any $n \in \mathbb{Z}$, consider the complex line bundle L_n given by

$$g_{\mathcal{U}\mathcal{V}} = (z/|z|)^n$$

where $z = x + iy$. Evaluate $\int_\Sigma c_1(L_n)$.

(4) Consider the following coordinate cover for $M = \mathbb{CP}^2$:

$$\mathcal{U} = \{[(1, u_2, u_3)] : (u_2, u_3) \in \mathbb{C}^2\} \ ,$$
$$\mathcal{V} = \{[(v_1, 1, v_3)] : (v_1, v_3) \in \mathbb{C}^2\} \ ,$$
$$\mathcal{W} = \{[(w_1, w_2, 1)] : (w_1, w_2) \in \mathbb{C}^2\} \ .$$

Let $N = \{[(x, y, 0)] : (x, y) \in \mathbb{C}^2 \setminus \{0\}\} \cong \mathbb{CP}^1$. Note that $M = \mathcal{W} \amalg N \cong \mathbb{C}^2 \amalg \mathbb{CP}^1$. In means that M can be regarded as a compatification of $\mathcal{W} \cong \mathbb{C}^2$ by adding a $\mathbb{CP}^1$, the space of complex directions in $\mathbb{C}^2$, at infinity.

(a) Construct a complex line bundle L over M by the following requirement: It has a global section $\underline{s} \in \Gamma(E)$, which is u_3 over $\mathcal{U}$, v_3 over $\mathcal{V}$, and 1 over $\mathcal{W}$ (with respect to some trivializations).

It follows that the zero locus of $\underline{s}$ is N.

(b) Construct a connection ∇ on L, and compute its curvature F^{∇}.

(c) Evaluate $\int_N c_1(L)$.

(d) In part (b), is it possible to choose ∇ such that F^{∇} only supports on a tubular neighborhood of N. Specifically,

$$\mathrm{supp}(F^{\nabla}) \cap \{[(w_1, w_1, 1)] : |w_1|^2 + |w_2|^2 < R^2\} = \varnothing$$

for some $R > 0$.

(5) Let A be a square-matrix valued 1-form, and denote $\mathrm{d}A + A \wedge A$ by F. Verify that

$$\mathrm{tr}(F \wedge F) = \mathrm{d}\left[\mathrm{tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A)\right] . \qquad (\dagger)$$

Remark. To construct the right hand side of $(\dagger)$, apply the argument on the invariance of the characteristic class in the de Rham cohomology. Specifically, consider $\mathrm{d} + tA$.