§I. heat Kernel estimates

<u>theorem 0</u> (Berard - Besson - Gallot. Invent. Math 85)

$r_0(M,g) = \inf\{ \text{Ricci}(u,u) : u$ unit tangent vector to $M \}$

$d(M,g) =$ diameter of (M,g)

$= \sup\{ d(x,y) : x,y \in M \}$

If $r_0^2 \cdot d \geq (n-1)\,\varepsilon\,\alpha^2$ for some $\alpha > 0$
$\varepsilon \in \{1, 0, -1\}$

Then, $\text{Vol}(M)\, k(t,x,x)$
$\leq Z_{S^n}(t/R^2)$ for some $R = R(n,\varepsilon,\alpha)$

unit sphere

<u>rmk</u> (i) $g \rightsquigarrow Rg \qquad \lambda \rightsquigarrow \frac{1}{R^2}\lambda$

$S^n(1) : \lambda_0, \lambda_1, \dots$

$S^n(R) : \frac{\lambda_0}{R^2}, \frac{\lambda_1}{R^2} \dots$

$$\Rightarrow Z_{S^n(R)}(t) = \sum_{i\geq 0} e^{-\frac{\lambda_0}{R^2}t} = Z_{S^n(1)}\left(\frac{t}{R^2}\right)$$

(ii) From the discussion last time, it remains to show $h_{S^n(R)}(\beta)$ is an Isoperimetric estimator of (M,g)

<u>defn</u> $\mathcal{M}_{n,k,D} = \{ (M^n,g) : r_0(M,g) \geq (n-1)k,\ d(M,g) \leq D \}$

$k \in \mathbb{R},\ D > 0$

<u>theorem 1</u> Fix (n, k, D). $\forall (M, g) \in \mathcal{M}_{n,k,D}$

i) $\lambda_j \geq c\, j^{2/n}$

ii) $N(\lambda) = \#\{ j : \lambda_j \leq \lambda \} \leq 1 + c\lambda^{n/2}$

iii) $\forall x \in M$, $\alpha \geq 0$

$$\sum_{j\geq 1} \lambda_j^{\alpha} \exp(-t\lambda_j)\, \varphi_j^2(x) \leq \frac{c}{Vol(M)}\, t^{-\frac{n+2\alpha}{2}}$$

Here, $c = c(n, k, D, \alpha)$

Pf: <u>step 1</u> If $k \geq 0$, $r_0 d^2 \geq 0$

If $k \leq 0$, $r_0 d^2 \geq (n-1) k D^2$

$\Rightarrow$ Theorem 0 applies

$$Z_M(t) = \int_{x\in M} k_M(t, x, x) \leq Vol(M) \sup_{x\in M} \{ k(t, x, x) \}$$

$$\leq Z_{S^n(1)}(t/R^2) = Z_{S^n(R)}(t)$$

$$Z_{S^n(R)}(t) = 1 + \sum_{j\geq 1} e^{-\lambda_j t}$$

For $t < 1$, by the parametrix construction.

$$Z_{S^n(R)}(t) \lesssim t^{-n/2}$$

For $t > 1$, we already know $\lambda_j \gtrsim j^{(S+\frac{n}{2})^{-1}}$

of $S^n(R)$

$\gtrsim j^{\frac{1}{n}}$

$$Z_{S^n(R)}(t)-1 = \sum_{j\geq 1} e^{-\lambda_j t} \lesssim \sum_{j\geq 1} e^{-j^{\frac{2}{n}} t}$$

λ_j of $S^n(R)$

$$\lesssim \int_1^\infty e^{-s^{\frac{2}{n}} t}\, ds \lesssim \int_1^\infty e^{-s^{\frac{2}{n}} t} s^{\frac{2}{n}-1}\, ds$$

$$\approx \int_1^\infty e^{-s^{\frac{2}{n}} t}\, ds^{\frac{2}{n}}$$

$$= e^{-t}$$

Hence. $Z_{S^n(R)}(t) - 1 \lesssim t^{-n/2} \quad \forall t$

step 2 $\quad Z_M(t) - 1 = \sum_{j\geq 1} e^{-\lambda_j t}$

$\Rightarrow \quad j \leq N(\lambda_j) - 1$

due to multiplicity

If $\lambda_i \leq \lambda_j \Rightarrow \lambda_i/\lambda_j \leq 1 \Rightarrow -\lambda_i/\lambda_j \geq -1$

(j: fixed)

$\Rightarrow e^{-\lambda_i/\lambda_j} \geq e^{-1}$

$\Rightarrow e \cdot e^{-\lambda_i \frac{1}{\lambda_j}} \geq 1$

$$\leq e \sum_{0<\lambda_i\leq\lambda_j} e^{-\lambda_i \frac{1}{\lambda_j}} \leq e\cdot\left(Z_M\left(1/\lambda_j\right) - 1\right)$$

$$\leq e\cdot\left(Z_{S^n(R)}\left(\tfrac{1}{\lambda_j}\right) - 1\right)$$

$$\lesssim \lambda_j^{n/2}$$

$\Rightarrow$ i) & ii)

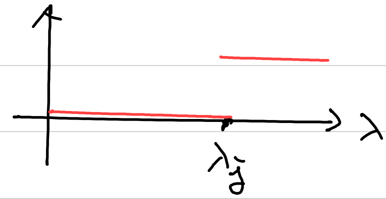
step 3 $\quad \forall x \in M$

Consider $d\mu_x = \sum_{j\geq 1} \varphi_j^2(x)\, \delta_{\lambda_j}$

weight $\quad$ delta distribution

a measure on $\mathbb{R}_{>0}$

$$\int_0^\lambda \delta_{\lambda_j} = \begin{cases} 1 & \lambda \geq \lambda_j \\ 0 & \lambda < \lambda_j \end{cases}$$



$$\Rightarrow \int_0^\lambda d\mu_x = \sum_{0<\lambda_j\leq\lambda} \varphi_j^2(x) =: \mu_x([0,\lambda])$$

$$\sum_{j\geq 1} \lambda_j^\alpha e^{-t\lambda_j} \varphi_j^2(x) = \int_0^\infty \underbrace{\lambda^\alpha e^{-t\lambda}}_{d} d\mu_x(\lambda)$$

$$\alpha\lambda^{\alpha-1}e^{-t\lambda} - t\lambda^\alpha e^{-t\lambda}$$

$$= \int_0^\infty \left(t\lambda^\alpha e^{-t\lambda} - \alpha\lambda^{\alpha-1}e^{-t\lambda}\right)\mu_x([0,\lambda])\, d\lambda$$

$$\leq \int_0^\infty (\lambda t+\alpha)\,\lambda^{\alpha-1} e^{-t\lambda}\left(\sum_{0<\lambda_j\leq\lambda}\varphi_j^2(x)\right) d\lambda$$

$$\leq e\cdot \sum_{0<\lambda_j\leq\lambda} e^{-\lambda_j \frac{1}{\lambda}} \varphi_j^2(x)$$

$$\leq e\cdot \sup_{x\in M}\left(k_M\left(\tfrac{1}{\lambda}, x, x\right) - \frac{1}{vol(M)}\right)$$

$$\leq \frac{e}{vol(M)}\left(Z_{S^n(R)}\left(\tfrac{1}{\lambda}\right) - 1\right) \lesssim \lambda^n$$

$$\lesssim \int_0^\infty (\lambda t+\alpha)\,\lambda^{\alpha-1} e^{-t\lambda}\lambda^n\, d\lambda \lesssim t^{-(\alpha+\frac{n}{2})}$$

#

§ II. embedding (M,g) into ℓ^2

(M,g) $\quad 0=\lambda_0 < \lambda_1 \le \lambda_2 < \cdots < \lambda_j < \cdots$

$\quad (\mathrm{Vol}(M))^{-1/2} = \varphi_0 \quad \varphi_1 \quad \cdots \quad \varphi_j$

<u>defn</u> Fix an orthonormal eigenbasis. let

$$\Psi_t : M \to \ell^2$$
$$x \mapsto \underbrace{\sqrt{2}\,(4\pi)^{n/2}}\, t^{\frac{n+2}{4}} \left\{ e^{-\lambda_j t/2} \varphi_j(x) \right\}_{j\ge 1}$$

will see the reason later

<u>rmk</u>
- φ_0 is dropped
- $\ell^2 \ni \{a_j\}_{j\ge1}$ with $\sum_{j\ge1} |a_j|^2 < \infty$

$$\sum_{j\ge1} \left| e^{-\lambda_j t/2} \varphi_j(x) \right|^2 = \sum_{j\ge1} e^{-\lambda_j t} \varphi_j(x)\varphi_j(x)$$
$$= k(t,x,x) - 1$$

- $h^1 \ni \{a_j\}_{j\ge1}$ with $\sum_{j\ge1} |a_j|^2 (1 + j^{\frac{2}{n}}) < \infty$

$\quad \wr\wr$

$\quad H_1(M)$ $\qquad$ Rellich lemma : $h^1 \xhookrightarrow{\text{cpt}} \ell^2$

<u>theorem 2</u> i) $\forall t>0$. $\Psi_t : M \to \ell^2$ is an embedding

ii) $\Psi_t^*(g_0) = g + \frac{t}{3}\left(\frac{1}{2} R(g)\cdot g - \mathrm{Ricci}(g)\right) + O(t^2)$ as $t \to 0^+$

standard metric on ℓ^2 (an ∞-dim'l vector space) pf: <u>step 1</u> $\quad \Psi_t(x) = \{ e^{-\lambda_j t} \varphi_j(x) \}_{j \ge 1}$

$$\| \Psi_{t'}(x') - \Psi_t(x) \|_{\ell^2}^2 = \sum_{j \ge 1} \left| e^{-\lambda_j \frac{t'}{2}} \varphi_j(x') - e^{-\lambda_j \frac{t}{2}} \varphi_j(x) \right|^2$$

$$= \sum_{j \ge 1} \Big(e^{-\lambda_j t'} \varphi_j(x') \varphi_j(x') - 2 e^{\lambda_j \frac{t+t'}{2}} \varphi_j(x) \varphi_j(x') + e^{-\lambda_j t} \varphi_j(x) \varphi_j(x) \Big)$$

$$= k(t', x', x') + k(t, x, x) - 2 k\left(\tfrac{t+t'}{2}, x, x'\right)$$

$\Rightarrow \Psi_t(x)$ is continuous

$\Rightarrow \psi_t(x)$ is continuous

$\Rightarrow \psi_t(M)$ is compact in ℓ^2

<u>step 2</u> $\quad \forall t > 0, \ \Psi_t(X)$ is injective

If NOT, $\exists\, x_0 \neq x_1 \in M$

with $\varphi_j(x_0) = \varphi_j(x_1) \quad \forall j$

But this cannot happen:

Choose $f \in C^\infty(M)$ with $\int f = 0$

$f(x_0) = 0, \quad f(x_1) = 1$

$\Rightarrow f(x) = \sum_{j \ge 1} a_j \varphi_j(x)$

Since $\varphi_j(x_0) = \varphi_j(x_1) \Rightarrow f(x_0) = f(x_1) \rightarrow\leftarrow$

Hence $\psi_t : M \to \ell^2$ is homeomorphic to its image

If $(d\psi_t)|_x(V) = 0$

for some $x \in M, \ V \in T_xM \setminus \{0\}$

$\Rightarrow d\varphi_j|_x(V) = 0$: component of $d\Phi_t$

$\Rightarrow df|_x(V) = 0 \quad \forall f \in C^\infty(M)$

$\Rightarrow V = 0 \quad \rightarrow\leftarrow$

Hence, Ψ_t is an embedding

<u>step 3</u> $\| d\Phi_t|_x(V)\|^2_{\ell^2} = \sum_{j\geq 1} e^{-\lambda_j t} | d\varphi_j|_x(V)|^2$

$= (d_s k)|_{(x,x)}(V, V)$

d_s on $f(x,y) \in M\times M$ is $d_x d_y f(x,y)$

$(T_xM \times T_yM \to \mathbb{R})$

$$k(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{dist^2(x,y)}{4t}\right)$$
$$\left(u_0(x,y) + u_1(x,y)\, t + \cdots\right)$$

<u>facts</u>. By the iteration construction on u's and the Jacobi field equation type computation:

- $d_x(dist^2(x,y))|_{(x,x)}(V) = 0 = d_y(dist^2(x,y))|_{(x,x)}(V)$
- $d_s(dist^2(x,y))|_{(x,x)}(V,V) = -2g_x(V,V)$
- $u_1(x,x) = R_1(x)/6$
- $u_0(x,x) = 1$
- $d_s u_0|_{(x,x)}(V,V) = -\frac{1}{6} Ricci_x(V,V)$

$$\Rightarrow (d_s k)\big|_{(x,x)}(V,V)$$

$$= \frac{1}{(4\pi t)^{n/2}} \left(\frac{1}{2t} g(V,V) \cdot \left(u_0(x,x) + t\, u_1(x,x) + O(t^2) \right) - \frac{1}{6} \mathrm{Ricci}_x(V,V) + O(t) \right)$$

$\divideontimes$

§. III choice of eigenbasis ?

The above construction depends on the choice of an orthonormal eigenbasis

When the eigenvalues are all simple.

the freedom is $\{\pm 1\}^{\mathbb{N}}$

In general, let μ_i be the eigenvalues. (without repeating)

$$0 = \mu_0 < \mu_1 < \mu_2 \cdots < \mu_i <$$
$$\qquad\quad E_1 \quad\ E_2 \qquad\quad E_i \leftarrow \text{the eigenspace}$$

freedom is $\prod_{i \geq 1} O(\dim E_i) \cong \mathcal{B}$ ← space of orthonormal eigenbasis

↖ orthogonal matrix

$\forall a \in \mathcal{B} \rightsquigarrow \psi_t^a$ the embedding in § II

$\forall a, b \in \mathcal{B}$ we can define

$$(d(a,b))^2 := \sum_{i \geq 1} \mu_i^{-N} \underbrace{d_{E_i}(a|_{E_i}, b|_{E_i})}_{\text{distance on } O(\dim E_i)}^2$$

If $N > \frac{n}{2}$, it converges

↖ fix a choice.

$$\left(\begin{array}{l} a|_{E_i}, b|_{E_i} \leftrightarrow P_i \in O(\dim E_i) \\ \qquad\qquad P_i = \mathbb{1} \Leftrightarrow a|_{E_i} = b|_{E_i} \\ \mathrm{dist}^2(a|_{E_i}, b|_{E_i}) = \| P_i - \mathbb{1} \|^2 \leftarrow \text{square sum of entries} \end{array} \right)$$

Now, we would like to compare the embedding for different $a \in \mathcal{B}$

it is convenient to drop the t-factor

<u>defn</u> Given $a = \{\varphi_j^a\} \in \mathcal{B}$

let $I_t^a(x) = (\mathrm{Vol}(M))^{\frac{1}{2}} \{ e^{-\lambda_j t/2} \varphi_j(x) \}_{j \geq 1}$

$$: M \to \ell^2$$

<u>rmk</u> $g \mapsto R^2 g$, $\lambda_j \longmapsto R^{-2} \lambda_j$

$$\varphi_j \longmapsto R^{-\frac{n}{2}} \varphi_j$$

$$\mathrm{Vol}(M) \mapsto R^n \mathrm{Vol}(M)$$

$$\Rightarrow I_t^a(x, M, R^2 g) = I_{\frac{t}{R^2}}^a(x, M, g)$$

<u>theorem 3</u> $I : \mathbb{R}_{>0} \times \mathcal{B} \times M \to \ell^2$

$$(t, a, x) \mapsto I_t^a(x)$$

is continuous.

In fact, $\| I_t^a(x) - I_s^b(y) \|_{\ell^2}^2$

$$\leq \mathrm{Vol}(M) \left(\begin{array}{l} k(t,x,x) + k(s,y,y) - 2k\left(\frac{t+s}{2}, x, y\right) \\ + 2\, d(a,b) \sqrt{k_N(t,x,x)\, k_N(s,y,y)} \end{array} \right)$$

the N in defining d

$$k_N(t,x,x) = \sum_{j \geq 1} \lambda_j^{N/2} e^{-\lambda_j t} \varphi_j^2(x)$$

Pf: $\| I_t^a(x) - I_s^b(y) \|_{\ell^2}^2$

$$= \mathrm{Vol}(M) \sum_{i\geq 1} \left| e^{-\lambda_i \frac{t}{2}} \varphi_i^a(x) - e^{-\lambda_i \frac{s}{2}} \varphi_i^b(y) \right|^2$$

$$\mathrm{Vol}(M)^{-1} \| I_t^a(x) - I_s^b(y) \|_{\ell^2}^2$$

$$= \sum_{j\geq 1} e^{-\lambda_j t} \varphi_j^a(x) \varphi_j^a(x) + e^{-\lambda_j s} \varphi_j^b(y) \varphi_j^b(y)$$

$$-2 e^{-\lambda_j \frac{t+s}{2}} \varphi_j^a(x) \varphi_j^b(y) \quad = \varphi_j^a(y) + (\varphi_j^b(y) - \varphi_j^a(x))$$

$$= k(t,x,x) + k(s,y,y) - 2k\left(\tfrac{t+s}{2}, x, y\right)$$

$$-2 \sum_{j\geq 1} e^{-\lambda_j \frac{s+t}{2}} \left(\varphi_j^a(x) \varphi_j^b(y) - \varphi_j^a(x) \varphi_j^a(y) \right)$$

sum over E_i part: $\{ j : \lambda_j = \mu_i \}$

$$\varphi_j^b(y) = P_j^{j'} \varphi_{j'}^a(y)$$

$= P|_{E_i}$

$$e^{-\mu_i \frac{s+t}{2}} \sum_{j,j'}^{\dim E_i} \varphi_j^a(x) \varphi_{j'}^a(x) \left(P_j^{j'} - \delta_j^{j'} \right)$$

$$\Rightarrow |\cdots| \leq e^{-\mu_i \frac{s+t}{2}} \mu_i^{+\frac{N}{2}} \left(\mu_i^{-\frac{N}{2}} d_{B*}(a|_{E_i}, b|_{E_i}) \right) \left(\sum_{E_i} (\varphi_j^a(x))^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{E_i} (\varphi_j^a(x))^2 \right)^{\frac{1}{2}}$$

Cauchy-Schwarz for $\sum_{j,j'} u_j P_{jj'} v_{j'}$

$$\left(\sum_{E_i} (\varphi_j^a(x))^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{E_i} (\varphi_j^a(x))^2 \right)^{\frac{1}{2}} = \sum_{E_i} (\varphi_j^a(x))^2$$

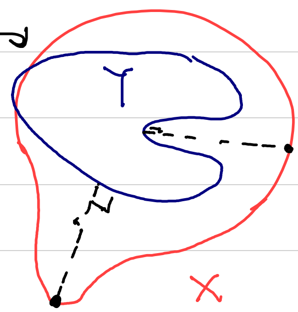
$\Rightarrow$ sum over i #

<u>Some terminology</u> Hausdorff distance

$X, Y \subset (M, d)$: metric space

$$HD(X,Y) = \max \{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(X, y) \}$$

$dist(X, Y) = 0$



y_0 ← achieve 2^{nd} sup

it measures the "farthest" distance

x_0 ← achieve 1^{st} sup

i) $X_\varepsilon := \bigcup_{x \in X} \{ z \in M : d(z, x) < \varepsilon \}$

$$HD(X,Y) = \inf \{ \varepsilon \geq 0 : X \subseteq Y_\varepsilon, Y \subseteq X_\varepsilon \}$$

If $HD(X,Y) = \delta$, $X \subseteq \overline{Y_\delta}$ and $Y \subseteq \overline{X_\delta}$

Hence, $HD(X,Y) = 0$ if and only if $\overline{X} = \overline{Y}$

ii) In general, HD is a pseudometric

On the set of all non-empty compact subsets, HD is a metric

Now, fix $t > 0$ $\forall a \in B(M)$

$$I_t^a : M \hookrightarrow \ell^2$$

$I_t^a(M) \subset \ell^2$ is a compact subset

$\Rightarrow \{ I_t^a(M) : a \in B(M) \}$ is a subset of $\mathcal{F}(\ell^2)$

where $\mathcal{F}(\ell^2) = \{$non-empty compact subsets of $\ell^2 \}$

$\rightsquigarrow$ Consider the distance between
$\{I_*^a(M) : a \in B(M)\}$ and
$\{I_*^b(M') : b \in B(M')\}$ subset

$(l^2, d) \rightsquigarrow (\mathcal{F}(l^2), HD)$

point here = compact subsets in l^2

$\rightsquigarrow$ Use the Hausdorff distance
of $(\mathcal{F}(l^2), HD)$

<u>defn</u> Define $d_*(M, M')$ to be

$$\max\left\{ \sup_{a \in B(M)} \inf_{b \in B(M')} HD(I_*^a(M), I_*^b(M')),\ \sup_{b \in B(M')} \inf_{a \in B(M)} HD(I_*^a(M), I_*^b(M')) \right\}$$

<u>theorem 4</u> Fix $*>0$. $d_*(M, M') = 0$ if and only if M and M' are isometric

pf: <u>step 1</u> Suppose that $d_*(M, M') = 0$

$$\Rightarrow \sup_{b \in B(M')} \inf \cdots = 0$$

$$\Rightarrow \forall b \in B(M') \quad \inf_{a \in B(M)} HD(I_*^a(M), I_*^b(M')) = 0$$

$\Rightarrow \exists a_\ell \in B(M)$ such that
$HD(I_\ell^{a_\ell}(M), I_*^b(M')) = 0$

Since $B(M)$ is compact, $a_\ell \to a \in B(M)$

Consider $HD(I_t^{a_\ell}(M), I_t^a(M))$

$\|I_t^{a_\ell}(x) - I_t^a(x)\|_{\ell^2}^2$ by theorem 3

$\leq Vol(M) \left(k(t,x,x) + \cancel{k(t,x,x) - 2k(t,x,x)} + 2d(a_\ell, a)\sqrt{k_0(t,x,x)} \right)$

Hence, their HD $\to 0$ as $\ell \to \infty$

$\Rightarrow HD(I_t^a(M), I_t^b(M')) = 0$

<u>step 2</u> Let $a \leftrightarrow \{\varphi_j\}_{j\geq 1} \subset C^\infty(M)$

$b \leftrightarrow \{\varphi_j'\}_{j\geq 1} \subset C^\infty(M')$

$\forall x \in M \quad \exists y \in M' \quad \ni I_t^a(x) = I_t^b(y')$

$\Leftrightarrow \sqrt{Vol(M)}\, e^{-\lambda_j t/2} \varphi_j(x)$

$= \sqrt{Vol(M')}\, e^{-\lambda_j' t/2} \varphi_j'(y)$

Similarly, $\forall y \in M'$, $\exists x \in M$

... as above ...

As we have shown that eigenbasis separates points, this $x \in M \underset{h_t}{\overset{f_t}{\rightleftarrows}} y \in M'$ correspondence is bijective (and continuous)

<u>step 3</u> diffeomorphism?

<u>lemma</u> $\forall x_0 \in M$, $\exists \varphi_{j_1}, \ldots, \varphi_{j_n}$ such that $\{\nabla\varphi_{j_1}|_{x_0}, \ldots, \nabla\varphi_{j_n}|_{x_0}\}$ span $T_{x_0}M$

$$\left(\begin{array}{l} \text{If NOT } \operatorname{span}\{\nabla\varphi_j|_{x_0} : j\geq 1\} \subsetneq T_{x_0}M \\ \Rightarrow C^\infty(M) \ni f = a_0 + \sum_{j\geq 1} a_j \varphi_j \\ \qquad\qquad \Rightarrow \nabla f|_{x_0} \subsetneq T_{x_0}M \qquad \rightarrow\leftarrow \end{array} \right)$$

From the lemma. consider

$$F : M \times M' \longrightarrow \mathbb{R}^n$$
$$(x, y) \longmapsto \left(\varphi_{j_i}(x) - e^{(\lambda_j - \lambda_j')\frac{t}{2}} \sqrt{\frac{Vol(M')}{Vol(M)}}\, \varphi'_{j_i}(y)\right)_{i=1,\dots,n}$$

$\Rightarrow F(h_t(y), y) \equiv 0$

Let $y_0 = f_t(x_0) \Rightarrow \partial_x F|_{x_0}$ is an isomorphism by the lemma

$\Rightarrow h_t(y)$ is smooth at y_0 by IFT

<u>step 4</u> $\sqrt{Vol(M)}\, e^{-\lambda_j t/2} \varphi_j(x) = \sqrt{Vol(M')}\, e^{-\lambda_j' t/2} \varphi_j'(y)$

$$0 = \sqrt{Vol(M)}\, e^{-\lambda_j \frac{t}{2}} \int_M \varphi_j(x)\, d\mu_x$$

$$= \sqrt{Vol(M')}\, e^{-\lambda_j' \frac{t}{2}} \underbrace{\int_M \varphi_j'(f_t(x))\, d\mu_x}_{= \int_{M'} \varphi_j'(y)\, a_t(y)\, d\mu'_y}$$

$a_t(y)\, d\mu'_y = h_t^* d\mu_x$

$\Rightarrow (a_t, \varphi_j')_{L^2(M')} = 0 \quad \forall j \geq 1$

$\Rightarrow a_t = \text{constant} = Vol(M) / Vol(M')$

$$Vol(M)\, e^{-\lambda_j t} (\varphi_j(x))^2 = Vol(M')\, e^{-\lambda_j' t} (\varphi_j'(y))^2$$

$$\Rightarrow \mathrm{Vol}(M)\, e^{-\lambda_j t} = \mathrm{Vol}(M)\, e^{-\lambda_j t} \int_M \varphi_j^2 \, d\mu_x$$

$$= \mathrm{Vol}(M')\, e^{-\lambda_j' t} \underbrace{\int_M (\varphi_j' \circ f_*)^2 \, d\mu_x}_{= \int_{M'} (\varphi_j')^2 a_t \, d\mu_y'}$$

$$= \mathrm{Vol}(M')\, e^{-\lambda_j' t} \cdot \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(M')}$$

$$\Rightarrow \lambda_j = \lambda_j' \quad \forall j \geq 1$$

$$\Rightarrow \mathrm{Vol}(M) = \mathrm{Vol}(M')$$ ← (later)

The above discussion implies that

$$\begin{tikzcd} (M,g) \arrow[r, bend left, "f_*"] & (M',g') \arrow[l, bend left, "h_*"] \arrow[r, "u"] & \mathbb{R} \end{tikzcd}$$

$$(\Delta_{(M',g')} u) \circ f_* = \Delta_{(M,g)} (u \circ f_*)$$

By considering the $\underbrace{\text{principal symbols}}_{= \text{2nd order derivative part}}$ of Δ

$\Rightarrow$ isometry #

§IV a pre-compactness property

<u>theorem 5</u> $\forall t > 0$, $\mathcal{M}_{n,\kappa,D}$ is d_t-precompact

(the d_t-completion is compact)

key: $h^1 \ni \{a_j\}_{j\geq 1} : \sum_{j\geq 1} (1 + j^{\frac{2}{n}}) |a_j|^2 < \infty$

$$\| I_t^a(x) \|_{h^1}^2 = \mathrm{Vol}(M) \cdot \sum_{j\geq 1} (1 + j^{\frac{2}{n}})\, e^{-\lambda_j t} \varphi_j^2(x)$$

theorem 1 i)

$$\lesssim \mathrm{Vol}(M) \sum_{j \geq 1} (1 + \lambda_j)\, e^{-\lambda_j t} \varphi_j^2(x)$$

$$\underset{\text{theorem 1 iii)}}{\lesssim} \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(M)} \left(t^{-n/2} + t^{-\frac{n+2}{2}}\right) \lesssim C_t(n,k,D)$$

$\Rightarrow \{ I_t^a(x) : x \in M \subset \mathcal{M}_{n,k,D}, a \in B(M) \}$

is bounded in h^1

By Rellich lemma, has compact closure K in ℓ^2

$\Rightarrow$ all the construction is based on $(K \subset \ell^2, d)$ ※

rmk Of course, the limits need not to be smooth manifold.

§ V. Weyl's Law.

1° $k(t,x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{d^2(x,y)}{4t}\right)$

$(u_0(x,y) + t \cdot \cdots)$

$$\sum_{j\geq 0} e^{-\lambda_j t} = \int_{x\in M} k(t,x,x)\, d\mu_x = \frac{1}{(4\pi t)^{\frac{n}{2}}} \mathrm{Vol}(M) + O\left(t^{-\frac{n}{2}+1}\right)$$

as $t \to 0$

$\Rightarrow \{\lambda_j\}_{j\geq 0}$ determines $\mathrm{Vol}(M)$

2° <u>Prop</u> (Karamata Tauberian theorem)

Let $d\mu(\lambda)$ be a positive measure on $\mathbb{R}_{\geq 0}$ such that $\int_0^\infty e^{-t\lambda}\, d\mu(\lambda) < \infty \quad \forall\, t > 0$

Suppose that $\lim\limits_{t \to 0} t^\alpha \int_0^\infty e^{-t\lambda}\, d\mu(\lambda) = C$ for some $\alpha, C > 0$

Then. $\lim\limits_{t\to 0} t^\alpha \int_0^\infty f(e^{-t\lambda})\, e^{-t\lambda}\, d\mu(\lambda)$

$$= \frac{C}{\Gamma(\alpha)} \int_0^\infty f(e^{-t})\, t^{\alpha-1} e^{-t}\, dt$$

for any $f \in C^0([0,1])$

pf: By Weierstrass. it suffices to show it for polynomials.

Consider $f(x) = x^k$

$$\lim_{t\to 0} t^\alpha \int_0^\infty \underbrace{e^{-tk\lambda}\, e^{-t\lambda}}_{= e^{-(k+1)t\lambda}}\, d\mu(\lambda) = C(k+1)^{-\alpha}$$

by assumption

$$\frac{C}{\Gamma(\alpha)} \int_0^\infty e^{-kt}\, t^{\alpha-1} e^{-t}\, dt = C(k+1)^{-\alpha}$$

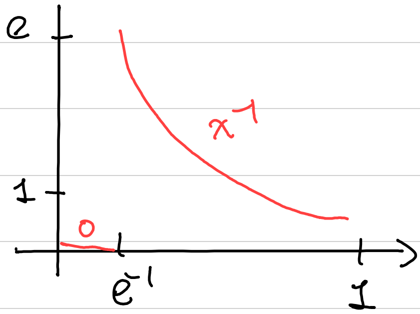
by definition/computation

3° $d\mu(\lambda) = \sum\limits_{j\geq 1} \delta_{\lambda_j} \Rightarrow \int_0^\infty e^{-t\lambda}\, d\mu(\lambda)$

$$= \sum_{j\geq 1} e^{-t\lambda_j}$$

$\alpha = \frac{n}{2}$

$$\Rightarrow \lim_{t \to 0} t^{\frac{n}{2}} \int_0^\infty e^{-t\lambda} \, d\mu(\lambda) = \frac{Vol(M)}{(4\pi)^{n/2}} = C$$



Let $f(x) = \begin{cases} x^{-1} & \text{on } [e^{-1}, 1] \\ 0 & \text{on } [0, e^{-1}] \end{cases}$

$$t^{\frac{n}{2}} \int_0^\infty f(e^{-t\lambda}) \, e^{-t\lambda} \, d\mu_\lambda \qquad e^{-t\lambda} \geq e^{-1}$$

$$t\lambda \leq 1$$

$$= t^{\frac{n}{2}} \, \#\{ j : \lambda_j \leq t^{-1} \} \qquad \lambda \leq t^{-1}$$

$$\Rightarrow \lim_{t \to 0} t^{\frac{n}{2}} \#\{ j : \lambda_j \leq t^{-1} \} = \frac{Vol(M)}{(4\pi)^{n/2}} \frac{2}{n} \frac{1}{\Gamma(n/2)}$$

$$\Rightarrow \lambda^{\frac{n}{2}} \#\{ j : \lambda_j \leq \lambda \} = \frac{Vol(M)}{(4\pi)^{n/2}} \frac{2}{n\Gamma(\frac{n}{2})} + o(1)$$

$$\#\{ j : \lambda_j \leq \lambda \} = \frac{Vol(M)}{(4\pi)^{n/2}} \frac{2}{n\Gamma(\frac{n}{2})} \lambda^{\frac{2}{n}} + o(\lambda^{\frac{2}{n}})$$

<u>rmk</u> One may also study the zeta function

$$\zeta(s) = \sum_{j \geq 0} \frac{1}{\lambda_j^s} \quad \text{for } s \in \mathbb{C}$$