TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 11

DUE MAY 12

(1) Let E be a vector bundle over M. Suppose that E carries a bundle metric (i.e. a fiberwise inner product), and a metric connection. Suppose that M carries a Riemannian metric. For a section ψ of E, the rough Laplacian is defined to be $\Box\,\psi = \sum_i (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})\psi$, where $\{e_i\}$ is an orthonormal frame of TM, and the smaller ∇ is the Levi-Civita connection for TM.

  (a) Check that $\Box\,\psi$ is independent of the choice of orthonormal frames.

  (b) Verify that
$$\Delta|\psi|^2 = 2\langle \Box\,\psi, \psi\rangle + 2|\nabla\psi|^2 \ ,$$
  where Δ is the Laplacian of (M, g) acting on scalar functions.

  (c) Given any smooth section ψ, show that on the open set where $\psi \neq 0$,
$$|\nabla\psi| \geq |\nabla|\psi|| \ .$$
  Remark. Then the inequality is true globally in a suitable sense.

(2) Suppose that M^n is an oriented, *minimal hyperplane* in $\mathbb{R}^n$. With a choice of unit normal, the second fundamental form is a section of $T^*M \otimes T^*M$. Denote the second fundamental form by $A = \sum_{1\leq i,j\leq n} h_{ij}\,\omega^i \otimes \omega^j$, where $\{\omega^i\}$ is the orthonormal coframe.

  (a) Since h_{ij} is symmetric, one may assume at any point p that $h_{ij} \overset{\text{at } p}{=} h_{ii}\delta_{ij}$. Prove that (at p)
$$|\nabla|A||^2 \leq n \sum_{i\neq j} h_{jj;i}^2 \ .$$

  (b) Prove that
$$|\nabla A|^2 \geq \left(1 + \frac{2}{n}\right) |\nabla|A||^2 \ .$$
  In other words, part (c) of (1) can be improved under this setting.