TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 10

DUE MAY 5

(1) Does there exists a map $f : (T^2, \mathrm{d}s^2_{\mathrm{std}})$ to some (M, g) which satisfies the harmonic map equation and whose image is a curve? If so, give an example. If not, prove there is no such an example.

(2) Examine the effect of the conformal change of the base metric for each terms in the Bochner formula of the energy density.

More precisely, let U be an open neighborhood of the origin of $\mathbb{R}^2$, with the metric $\eta_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b$. For an harmonic map $f : (U, \eta) \to (M, g)$, its energy density obeys

$$\Delta_\eta e(f) = |\nabla^\eta \mathrm{d}f|^2 + 2K_\eta \cdot e(f) - 2K_M(\mathrm{d}f) \cdot (a(f))^2 \ .$$

What happens if one endow U with the metric $e^{2\rho}\eta$ for some $u \in C^\infty(U, \mathbb{R})$.

(3) Consider S^2 with the standard round metric of Gaussian curvature 1. Provide a sequence of harmonic maps $f_k : S^2$ to some compact (M, g) such that

- f_k is non-trivial;
- $E(f_k) \leq E_0$ for some $E_0 > 0$;
- f_k admits no C^0-convergence subsequence;
- f_k convergence on any compact subset of the complement of a point.

The limit, f_∞, shall be a harmonic map from $\mathbb{R}^2$. What is f_∞ in your example?