TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 09

DUE APRIL 28

(1) Consider the $T^2 = S^1 \times S^1$ with the standard product metric.

  (a) For $\mathbb{R}^2$-valued function $\psi = (u, v)$, let $\mathcal{L}(u, v) = (\partial_x u - \partial_y v, \partial_y u + \partial_x v)$. Explain that $\mathcal{L}$ is a (first order) elliptic operator[1].

  (b) Show by a *direct computation* that $||\psi||_{L^2_1} \leq ||\mathcal{L}(\psi)||_{L^2} + ||\psi||_{L^2}$.

(2) Let (M^n, g) be a compact, oriented Riemannian manifold. Let $\alpha = \alpha_i \mathrm{d}x^i$ be a 1-form on M.

  (a) Suppose that α is harmonic. That is to say, $\mathrm{d}\alpha = 0 = \mathrm{d}^*\alpha$, where $\mathrm{d}^*\alpha$ is the function given by $\mathrm{tr}(\nabla\alpha) = g^{ij}\alpha_{i;j}$. Note that $\mathrm{d}\alpha = 0$ is equivalent to that $\alpha_{i;j} = \alpha_{j;i}$ for any i, j. Derive the Bochner formula for $\Delta|\alpha|^2$.

  (b) Identify a curvature condition for M under which there is no non-trivial harmonic 1-forms.
  Remark: As a consequence of Hodge theory, the first Betti number of M vanishes.

[1] Find a definition by yourself