TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 08

DUE APRIL 21

(1) Let g be a Riemannian metric on a surface Σ. Show that the Gaussian curvature of $e^{2w}g$ is given by

$$K_w = e^{-2w}(K - \Delta w) \ .$$

(2) Let (M, g) be a Riemannian manifold, and ∇ be it Levi-Civita connection. For a tensor $S = S_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$, its covariant derivative is $\nabla S = S_{ij;k}\,\mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j$. In other words, $S_{ij;k} = (\nabla_{\partial_k} S)(\partial_i, \partial_j)$. Similarly, $S_{ij;k\ell} = S_{ij;k;\ell} = (\nabla_{\partial_\ell}(\nabla S))(\partial_k, \partial_i, \partial_j)$. This notations is also defined in the same way for tensors of other types.

(a) For a vector field of compact support, $V = V^i \partial_i$. Verify that $\int V^i_{;i}\mathrm{d}\mu = 0$ where $\mathrm{d}\mu$ is the volume form of g.

(b) If S is symmetric, $S_{ij} = S_{ji}$, verify that $S_{ij;k} = S_{ji;k}$.

(c) The connection Laplacian, also known as the rough Laplacian, is defined to be $\Box S = g^{k\ell} S_{ij;k\ell}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$. Note that the trace of S is $\mathrm{tr}(S) = g^{ij}S_{ij}$. Verify that $\Delta\,\mathrm{tr}(S) = \mathrm{tr}(\Box S)$.