TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 06

DUE MARCH 31

For two (separable) Banach spaces E_1 and E_2, let $\mathscr{L}(E_1, E_2)$ be the space of all bounded linear operators, and let $\mathscr{F}(E_1, E_2)$ be the space of all Fredholm operators.

(1) Prove the $\mathscr{F}(E_1, E_2)$ is an open subset of $\mathscr{L}(E_1, E_2)$, and the Fredholm index is locally constant on $\mathscr{F}(E_1, E_2)$.
Hint: For any $L \in \mathscr{F}(E_1, E_2)$, one has $E_1 \cong E_0 \oplus \ker(L)$ and $E_2 \cong E_0 \oplus \mathrm{coker}(L)$, where $E_0 \cong \mathrm{range}(L)$. You may study nearby maps by using this decomposition.

(2) Denote by E_1^* and E_2^* the space of bounded linear functionals on E_1 and E_2, respectively. Recall that for any $T \in \mathscr{L}(E_1, E_2)$, it naturally induces $T^* \in \mathscr{L}(E_2^*, E_1^*)$ with the same operator norm as that of L.
Prove that if T has closed range, then $(\mathrm{coker}(T))^* \cong \ker(T^*)$.

(3) Prove that for any compact operator $K \in \mathscr{L}(E, E)$, that operator $\mathbf{I}_E + K \in \mathscr{L}(E, E)$ is Fredholm. Also, determine its index.

(4) Prove that $L \in \mathscr{F}(E_1, E_2)$ if and only if there exists some $R \in \mathscr{L}(E_2, E_1)$ such that
$$R \circ L - \mathbf{I}_{E_1} \quad \text{and} \quad L \circ R - \mathbf{I}_{E_2} \quad \text{are compact operators.}$$