TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 05

DUE MARCH 24

Sard's theorem says that for a smooth map F, its critical values are of measure zero.

(1) Let M be a smooth n-dimensional manifold. Suppose there is a smooth function $f : M \to \mathbb{R}$.
  (a) If p is a critical point of f, show that its Hessian, $\mathrm{Hess}_p(f)$, is a well-defined bilinear map from $T_pM \times T_pM$ to $\mathbb{R}$.
  (b) The function f is called a *Morse function* if $\mathrm{Hess}_p(f)$ is non-degenerate for every critical point p. Show that critical points of a Morse function are isolated.

(2) Let $U \subset \mathbb{R}^n$ be an open subset, and $f : U \to \mathbb{R}$ be a smooth function. For any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, define $f_a(x)$ to be $f(x) + \sum_{i=1}^n a_i x^i$. Prove that for a generic[1] $a \in \mathbb{R}^n$, the function f_a is a Morse function on U.

(3) For a smooth submanifold $M \subset \mathbb{R}^N$ and a smooth function $f : M \to \mathbb{R}$, prove that

$$f_a(x) = f(x) + \sum_{j=1}^{N} a_j x^j$$

is a Morse function on M for a generic $a \in \mathbb{R}^N$.

In particular, one can apply it to $f = 0$. This exercise asserts that the projection onto a generic direction is a Morse function.

[1] Here, being generic means except on a set of measure zero. Later on, it will has other definition.